Construction of Circular Distant Divisor Graphs

Janagam Veeranjaneyulu[1*], Peruri Lakshmi Narayana Varma[2]

Abstract

Let H be a graph with n vertices and e edges. A circular path P between two distant vertices is called a circular distant divisor path, if circular length of P divides e, the number of edges in the graph. We can construct a new graph from H consisting of the vertex set same as that of H and two vertices are adjacent if they have a circular distant divisor path in H. This new graph is called circular distant divisor graph of H and is denoted by $CD(H)$. In this article, constructions have been done, regarding circular distant divisor graphs of some standard graphs such as path, cycle, star, wheel, complete and complete bipartite graphs. Moreover, family of corona graphs was studied. When this circular distant divisor graphs concept was applied to the corona graphs, it was observed that sometimes the new graph was also a corona graph and sometimes it was not a corona graph.

KeyWords: circular distance, circular distant divisor path, circular distant divisor graph, antipodal graph, corona graph

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Introduction

Throughout this article we consider finite, undirected, connected graphs without multiple edges and loops.

Let $H=(V,E)$ be a finite graph. Let r,s be two vertices of H. The circular distance between these vertices is the sum of geodesic and detour distances. This distance is denoted by $cir(r,s)$. The circular distance in graphs has a special significance. It is useful, for example, in vehicle to vehicle communication, automatic map generation of user in an unknown places etc. Thus one can reduce the journey time and fuel consumption of vehicles etc [1].

In network theory, circular antipodal graphs are useful to connect or disconnect the communication between specific vertices [2-4]. Number theory has a strong connection with Graph theory [5-7]. Using the divisibility concept in number theory, the followingconcept as introduced.

A path P between two vertices of H is called a distinct divisor path, if length of P, $l(P)$, is a divisor of number of edges in H. Using this concept, new classes of graphs, like distinct divisor graphs, detour distant divisor graphs, distant divisor graph of subdivision of graphs have been introduced [8, 9].

In the present article, we introduced a new concept called Circular distant divisor graphs by considering circular distant divisor path. A path is said to be circular distant divisor path if the length of the path divides the number of edges in H. In the following, first we computed the circular distant divisor graphs of some families of standard graphs such as complete graph, cycle graph, wheel graph, path graph, complete bipartite graph etc.

In the next section, we studied the circular distant divisor graph of a double star graph. To our surprise, these graphs are quite different for $St_{n,n,1}$, when is a prime and not a prime number. Next we studied relation between circular distant divisor graphs and circular antipodal graphs and obtained some results. Using this circular distant divisor graph, some mutations have been carried out regarding corona graphs of some standard graphs, like complete graph, cycle graph, path graph, star graph, double star graph etc.

Corresponding author: Janagam Veeranjaneyulu
Address: [1,2]Department of Mathematics, School of applied Science & Humanities, Vignan's Foundation for Science, Technology & Research, Vadlamudi - 522213, Guntur, India
E-mail: jangam.vfstr@gmail.com, plnvarma@gmail.com

Circular distant divisor graphs

In this part we introduce the concept of circular distant divisor graphs. For reader convenience we look back at some basic definitions.

Definition 1. Let H be a graph with n vertices and e edges. A geodesic path, P, between two vertices is called the distant divisor path if length of P divides the number of edges e.

Definition 2. Let H be a graph with n vertices and e edges. A detour path, P is called detour distant divisor path if length of P is divisor of e.

Definition 3. The circular distance between two vertices r, s of H is defined as the sum of geodesic distance and detour distance. This is denoted by $cir(r, s)$.

Now we introduce the concept of circular distant divisor graph as follows.

Definition 4. Let H be a graph. The circular path P is called a circular distant divisor path, if $l^C(\text{P})$ is a divisor of number of edges e.

Example 5. Consider the graph H, shown in Figure 1, with 6 vertices and 8 edges. Look at the vertices r_1 and r_4. Clearly $r_1 - r_4$ is the geodesic path and the path $r_1 - r_2 - r_3 - r_4$ is a detour path. Thus the circular path is of length is 4.

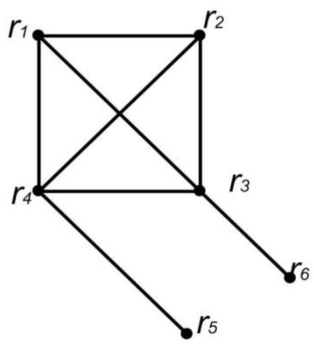

Figure 1. Graph H

As $e = 8$ and 4 divides 8, this circular path is clearly a circular distant divisor path.

Next, we define circular distant divisor graph of a graph.

Definition 6. Let H be a graph with n vertices and e edges. The circular distant divisor graph, $CD(H)$, of the graph H has the vertex set $V = V(H)$ and two vertices in $CD(H)$ are adjacent if there exists a circular distant divisor path in H.

The following example illustrates the concept of circular distant divisor graph of a graph.

Example 7. For the graph H, consider in Example 5, the circular distant divisor graph $CD(H)$ is as shown below. Consider the vertices r_5 and r_6, clearly $r_5 - r_4 - r_3 - r_6$ is the geodesic path and the $r_5 - r_4 - r_2 - r_1 - r_3 - r_6$ is detour path. Thus $l^C(P) = 8$ and it is a divisor of number of edges of H. So there exist a circular distant divisor path from r_5 to r_6. Similarly, it can be checked that all edges of H are in $CD(H)$. Thus $CD(H)$ is as shown in Figure 2.

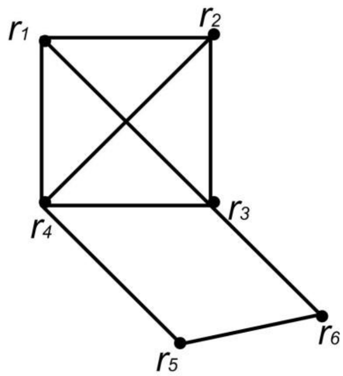

Figure 2. Graph $CD(H)$

Remark 8. From the above Examples 5 and 7, it is clear that $H \subseteq CD(H)$. But in general this need not be true, as shown in Example 29, below.

Circular distant divisor graph of some classes of graphs

In this section we construct the circular distant divisor graphs of some classes of graphs. We begin with circular distant divisor graph of a complete graph.

Theorem 9. Let K_n be the complete graph with $n\ (\geq 3)$ vertices. Then we have

$$CD(K_n) \cong \begin{cases} \overline{K}_n & \text{if } n \text{ is even} \\ K_n & \text{if } n \text{ is odd} \end{cases}$$

Proof. The complete graph K_n is a graph whose vertices can be listed in order as $r_1, r_2, \cdots, r_n$ so that the number of edges is $\binom{n}{2}$. Observe that

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$cir(r_i, r_j) = n$ for all $i \neq j.$

Now, we complete the proof in two cases, namely when n is even and odd.

Case 1 n even.

When n is even, n is not a divisor of $\binom{n}{2}$ and hence $cir(r_i, r_j)$ is not a divisor of number of edges, for all $i, j.$ Thus there is no circular distant divisor path between any two vertices of K_n. Thus in $CD(K_n)$, each vertex is an isolated vertex. Hence $CD(K_n) \cong \overline{K}_n$

Case 2 n odd.

When n is odd, n is a divisor of $\binom{n}{2}$ and hence $cir(r_i, r_j)$ is divisor of number of edges. Thus there is a circular distant divisor path between any two vertices of K_n. Thus, in $CD(K_n)$, any two vertices are adjacent. Hence $CD(K_n)$ is nothing but the complete graph K_n.

Next we will construct circular distant divisor graph of a cycle graph.

Theorem 10. Let C_n be the cycle graph with $n\ (\geq 3)$ vertices. Then the circular distant divisor graph of C_n is isomorphic to the complete graph K_n.

Proof. The cycle graph C_n is a graph whose vertices can be listed in order $r_1, r_2, \cdots, r_n$ so that the number of edges is $n.$ Now $cir(r_i, r_j) = n,$ for all $i \neq j.$ Then clearly $cir(r_i, r_j)$ divides number of edges $n.$ Then there exists a circular distant divisor path between any two vertices. Thus in $CD(C_n)$ each vertex is adjacent to every other vertex. Hence $CD(C_n) \cong K_n.$

Open problem 1. From Theorem 9 and 10, we can observe that, for any odd integer $n \geq 3$ we have $CD(K_n) \cong CD(C_n)$ but $K_n \ncong C_n$. Thus characterize all graphs for which $CD(H_1) \cong CD(H_2)$ implies $H_1 \cong H_2$.

Next, consider the class of wheel graph.

Theorem 11. Let $W_{1,n}$ be the wheel graph with $n+1$ vertices for $n \geq 3$. Then we have $CD(W_{1,n})$ is a discrete graph with $n+1$ vertices.

Proof. Let $W_{1,n}$ be a wheel graph with vertex set $\{r_0, r_1, r_2, \cdots, r_n\}.$ Without loss of generality, assume that r_0 is adjacent to all other vertices. The number of edges in $W_{1,n}$ is $2n$.

Consider the vertices $r_0, r_i, r_j \in W_{1,n},$ where r_i, r_j are two arbitrary vertices. Then $cir(r_0, r_i) = n+1.$ If r_i, r_j are adjacent then $cir(r_i, r_j) = n+1$ and if r_i, r_j are not adjacent then $cir(r_i, r_j) = n+2.$ None of these circular distances divide the number of edges $2n.$ Hence there is no circular distant divisor path between any two vertices of $W_{1,n}$. Thus in $CD(W_{1,n})$ each vertex is an isolated vertex. Hence $CD(W_{1,n}) \cong \overline{K}_{n+1}.$

Next, we look at the circular distant divisor graph of path graph.

Theorem 12. Let P_n be the path graph with n vertices, where $n \geq 4$. Then we have

$$CD(P_n) \cong \begin{cases} \overline{K}_n & \text{if } n \text{ is even} \\ \text{cyclic graph} & \text{if } n \text{ is odd} \end{cases}.$$

Further $CD(P_2) \cong \overline{K}_2$ $CD(P_3) \cong P_3.$

Proof. The path graph P_n is a graph whose vertices can be listed in order as $r_1, r_2, \cdots, r_n$ such that the edges are $\{r_i r_{i+1}\}$ where $i = 1, 2, 3, \cdots, n-1.$ Hence the number of edges is $n-1.$ There is a unique path between any two vertices and hence $D(r, s) = d(r, s).$

Now for $n = 2,$ we have $e = 1$ and $cir(r_i, r_j) = 2,$ so there can not be circular distant divisor path. Hence $CD(P_2) \cong \overline{K}_2.$

Now for $n = 3,$ we have $e = 2$ and $cir(r_i, r_j) = 2 \text{ or } 4.$ So there exists a circular distant divisor path when $cir(r_i, r_j) = 2,$ i.e., between r_i and $r_{i+1}.$ Thus $CD(P_3) \cong P_3.$

Now let us construct the graph $CD(P_n)$ for $n \geq 4.$ this will do in two cases, namely, n is even and odd.

Case 1 Suppose n is even.

In this case there exists $n-1,$ odd number of edges

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in P_n and $cir(r_i, r_j) = 2|j-i|$ is an even. So there is no circular distant divisor path between r_i and r_j for all $i, j, i \neq j$. Hence $CD(P_n) \cong \overline{K}_n$.

Case 2 Suppose n is odd for $n \geq 5$.

In this case, let the vertices of P_n be $r_1, r_2, \cdots, r_{\frac{n-1}{2}}, r_{\frac{n+1}{2}}, r_{\frac{n+3}{2}}, \cdots, r_{n-1}, r_n$ and the number of edges, $n-1$ even. Further $cir(r_i, r_j) = 2(j-i)$ is always an even number for all $1 \leq i < j \leq n$. Clearly 2 and $n-1$ divides number of edges.

As 2 divides $n-1$ there exists a circular distant divisor path between the vertices r_i, r_{i+1} for $1 \leq i \leq n-1$.

Next, since $n-1$ is a divisor of itself, there exist a circular distant divisor path between any two vertices r_i, r_j such that $\frac{n-1}{2} = (j-i)$ for all $1 \leq i < j \leq n$. Thus the circular distant divisor paths are from $r_1 - r_{\frac{n+1}{2}}, r_2 - r_{\frac{n+3}{2}}, \cdots, r_{\frac{n-1}{2}} - r_{n-1}, r_{\frac{n+1}{2}} - r_n$.

Further if any integer m, other than 2 and $n-1$, divides $n-1$, there exist a circular distant divisor path between any two vertices r_i, r_j such that $\frac{m}{2} = (j-i)$ for all $1 \leq i < j \leq n$. Thus some of the circular distant divisor paths are of the from $r_1 - r_{\frac{m+2}{2}}, r_2 - r_{\frac{m+4}{2}}, \cdots, r_{m-1} - r_{\frac{3m-2}{2}}, r_m - r_{\frac{3m}{2}}, r_{m+1} - r_{\frac{3m+2}{2}}, \cdots, r_{\frac{2n-m-2}{2}} - r_{n-1}, r_{\frac{2n-m}{2}} - r_n$.

Next, we claim that $CD(P_n)$ contains cycle of length $\frac{n+1}{2}$ and $m+1$ (provided m divides $n-1$). We can prove that the claim as follows. Clearly we can see that $CD(P_n)$ has the following cycles of length $\frac{n+1}{2}$:

$r_1 - r_2 - r_3 \cdots r_{\frac{n-3}{2}} - r_{\frac{n-1}{2}} - r_{\frac{n+1}{2}} - r_1$,

$r_2 - r_3 - r_3 \cdots - r_{\frac{n-3}{2}} - r_{\frac{n-1}{2}} - r_{\frac{n+1}{2}} - r_{\frac{n+3}{2}} - r_2$ etc.,

Next, we have the cycles of length $m+1$ as follows:

$r_1 - r_2 - \cdots - r_{\frac{m+2}{2}} - r_{\frac{m+4}{2}} - r_{\frac{m+6}{2}} - \cdots - r_1$,

$r_2 - r_3 - \cdots - r_{\frac{m+2}{2}} - r_{\frac{m+4}{2}} - r_{\frac{m+6}{2}} - \cdots - r_2$ etc.,

Thus we conclude that $CD(P_n)$, is a cyclic graph.

Next, we construct the circular distant divisor graph of star graph.

Theorem 13. Let $St_{n,1}$ be the star graph with $n+1$ vertices. Then we have

$$CD(St_{n,1}) \cong \begin{cases} \overline{K}_{n+1} & \text{if } n \equiv 1 \pmod 2 \\ K_{n+1} & \text{if } n \equiv 0 \pmod 4 \\ St_{n,1} & \text{if } n \equiv 0 \pmod 2 \text{ but } n \not\equiv 0 \pmod 4 \end{cases}$$

Proof. Let $\{r_0, r_1, r_2, \cdots, r_n\}$ be the vertex set of the star graph $St_{n,1}$ with r_0 adjacent to all other vertices. Clearly this graph has n edges. Further $d(r_0, r_i) = 1$ and $D(r_0, r_i) = 1$; $d(r_i, r_j) = 2$, $D(r_i, r_j) = 2$. Hence $cir(r_0, r_i) = 2$ and $cir(r_i, r_j) = 4$.

Now let us construct $CD(St_{n,1})$. This we will do in three cases as follows:

Case 1 Suppose n is odd

In this case, clearly the circular distances $cir(r_0, r_i) = 2$ and $cir(r_i, r_j) = 4$ are not divisors of n, the number of edges. Thus there is no circular distant divisor path between any two vertices. Thus $CD(St_{n,1}) \cong \overline{K}_{n+1}$.

Case 2 Suppose n is a multiple of 4.

In this case, both the circular distances $cir(r_0, r_i) = 2$ and $cir(r_i, r_j) = 4$ are divisors of n, the number of edges. Hence there exists a circular distant divisor path between any two vertices. Thus $CD(St_{n,1}) \cong K_{n+1}$

Case 3 Suppose n is a multiple of 2 but not a multiple of 4.

In this case, $cir(r_0, r_i) = 2$ is the only divisor of n, hence there exists circular distant divisor path between r_0 and any $r_i, 1 \leq i \leq n$. Hence $CD(St_{n,1}) \cong St_{n,1}$.

Next, we look at the complete bipartite graph $K_{m,n}$. First we deal with the case where $m < n$. $m = n$ case will be dealt separately.

Theorem 14. Let $K_{m,n}$ be the complete bipartite graph with $m+n$ vertices and $m < n$. Then we have

$$CD(K_{m,n}) \cong \begin{cases} K_{m+n} & \text{if } mn \text{ is multiple of } 2m \text{ and } 2m+2 \\ K_m + \overline{K}_n & \text{if } mn \text{ is multiple of } 2m \text{ but not of } 2m+2 \\ K_n \cup \overline{K}_m & \text{if } mn \text{ is multiple of } 2m+2 \text{ but not of } 2m \\ \overline{K}_{m+n} & \text{if } mn \text{ is not multiple of both } 2m \text{ and } 2m+2 \end{cases}$$

Proof. In the complete bipartite graph, $K_{m,n}$, the vertex set can be split into two sets $U=\{r_1, r_2, \cdots, r_m\}$ and $V=\{s_1, s_2, \cdots, s_n\}$ such that each edge will have one end vertex in U and the other in V. This graph has $m+n$ vertices and mn edges.

In $K_{m,n}$, we observe, that (see theorem 3.6 in [1]).

$cir(r_i, r_j) = 2m,$ for $1 \le i < j \le m.$

$cir(r_i, s_j) = 2m,$ for $1 \le i \le m, 1 \le j \le n.$

$cir(s_i, s_j) = 2m+2,$ for $1 \le i < j \le n$

Thus to construct the circular distant divisor graph, we need to check that the numbers $2m$ and $2m+2$ are divisors of mn or not.

Case 1 Suppose mn is a multiple of both $2m$ and $2m+2.$

In this case, clearly $2m$ and $2m+2$ divides $mn.$ So there exists a circular distant divisor path between any two vertices in $CD(K_{m,n}).$ Hence $CD(K_{m,n}) \cong K_{m+n}.$

See Example 15, below.

Case 2 Suppose mn is a multiple of $2m$ but not $2m+2.$

In this case, clearly $2m$ is divisor of mn but not $2m+2.$ So there exists a circular distant divisor path from r_i to r_j and from r_i to $s_j.$ There is no circular distant divisor path from s_i to s_j for all $i, j,$ $i \ne j.$ Hence $CD(K_{m,n}) \cong K_m + \overline{K}_n.$

See Example 16, below.

Case 3 Suppose mn is a multiple of $2m+2$ but not $2m$

In this case, clearly $2m+2$ is a divisor of $mn.$ So there exists a circular distant divisor path from s_i to s_j for all $i, j, i \ne j$ and all r_i's are isolated vertices. Hence $CD(K_{m,n}) \cong K_n \cup \overline{K}_m.$

See Example 17, below.

Case 4 Suppose mn is a not multiple of both $2m$ and $2m+2.$

In this case, clearly $2m$ and $2m+2$ does not divide $mn.$ So there does not exist circular distant divisor path between any pair of vertices. So in $CD(K_{m,n})$ each vertex is an isolated. Hence $CD(K_{m,n}) \cong \overline{K}_{m+n}.$

See Example 18, below.

Example 15. For $K_{2,6}$, $CD(K_{2,6}) \cong K_{2+6} \cong K_8.$

Example 16. For $K_{3,6}$, $CD(K_{3,6}) \cong K_3 + \overline{K}_6.$

Example 17. For $K_{2,3}$, $CD(K_{2,3}) \cong K_3 \cup \overline{K}_2$

Example18. For $K_{2,5}$, $CD(K_{2,5}) \cong \overline{K}_{2+5} \cong \overline{K}_7.$

Next, we have the following.

Theorem 19. Let $K_{m,m}$ be the complete bipartite graph with $2m$ vertices. Then we have

$$CD(K_{m,m}) \cong \begin{cases} \overline{K}_{2m} & \text{if } m \text{ is odd} \\ K_{2m} & \text{if } m \text{ is even} \end{cases}$$

Proof. In a complete bipartite graph, $K_{m,m}$, the vertex set can be split into two sets $U=\{r_1, r_2, \cdots, r_m\}$ and $V=\{s_1, s_2, \cdots, s_m\}$ so that each edge will have one end vertex in U and the other in $V.$ Note that in $K_{m,m}$, we have $2m$ vertices and m^2 edges. Further observe that in $K_{m,m}$ we have $cir(r_i, r_j)$ $cir(r_i, s_j) = cir(s_i, s_j) = 2m,$ for $1 \le i, j \le n$ (see theorem 3.6 in [1]).

To construct the circular distant divisor graph, we need to check that only the number $2m$ is divisor of m^2 or not.

The cases when m is odd and m is even will be dealt separately.

Case 1 Suppose m is odd.

In this case, $2m$ does not divide $m^2.$ So there does not exist circular distant divisor path between any two vertices in $CD(K_{m,m})$. Thus all the $2m$ vertices are isolated. Hence $CD(K_{m,m}) \cong \overline{K}_{2m}$.

Case2 Suppose m is even

In this case, $2m$ divides $m^2.$ So there exists circular distant divisor path between any two vertices in $CD(K_{m,m})$. Hence $CD(K_{m,m}) \cong K_{2m}$.

Circular distant divisor graphs of double star graph

In this section, we construct the circular distant divisor graphs of double star graphs. First we construct $CD(St_{n,n,1})$ when n is a prime number. Next, we deal with non-prime numbers. We begin with

Theorem 20. Let $St_{p,p,1}$ be the double star graph, where p is a prime number. Then $CD(St_{p,p,1})$ is isomorphic to $St_{p,p,1}$ for $p>3.$ For $p=3,$ we have $CD(St_{3,3,1})$ is the graph having vertex set same as that of $St_{3,3,1}$ and edge set consisting of all the edges of $St_{3,3,1}$ and all the edges joining degree 2 vertex and the pendent vertices. For $p=2, CD(St_{2,2,1})$ is nothing but $\overline{K}_4$.

Proof. Let P be a prime number greater than 3 and consider the double star graph, $St_{p,p,1}.$ Suppose that the vertex set of $(St_{p,p,1})$ is $\{r_0\}\cup\{r_1, r_2, \cdots, r_p\}\cup\{s_1, s_2, \cdots, s_p\}$ and edge set is $\{e_1, e_2, \cdots, e_p\}\cup\{f_1, f_2, \cdots, f_p\}$ where $e_i = r_0 r_i$ and $f_i = r_i s_i$. Thus in this graph number of vertices is $2n+1$ and that of edges is $2n.$

Observe that in this graph various circular distances are as follows:

$cir(r_0, r_i) = cir(r_i, s_i) = 2, \forall i$

$cir(r_0, s_i) = cir(r_i, r_j) = 4 \ \ \forall i \neq j.$

$cir(r_i, s_j) = 6$ and $cir(s_i, s_j) = 8 \ \ \forall i \neq j.$

Among these distances only 2 divides the number of edges, because n is prime greater than 3. Thus there exist circular distant divisor paths from r_0 to r_i and from r_i to $s_i.$ Further we do not have circular distant divisor paths between other edges (as n is prime). Thus $CD(St_{p,p,1}) \cong St_{p,p,1}$ for $n>3.$

We observe that in $St_{3,3,1},$ various circular distances are given by

$cir(r_0, r_i) = cir(r_i, s_i) = 2 \ \forall i$

$cir(r_0, s_i) = cir(r_i, r_j) = 4 \ \ \forall i \neq j.$

$cir(r_i, s_j) = 6$ and $cir(s_i, s_j) = 8 \ \ \forall i \neq j$

From the above distances, we can see that only 2 and 6 divide 6 which is the number of edges of $St_{3,3,1}.$ Thus in $CD(St_{3,3,1}),$ r_i and s_j are adjacent for $i, j = 1, 2, 3$ and it also contains all the edges of $St_{3,3,1}$. Then $E(CD(St_{3,3,1}))$ consists of all edges of $St_{3,3,1}$ and all the edges joining degree 2 vertices and the pendent vertices. Thus $CD(St_{3,3,1})$ is a (7, 12) graph shown below.

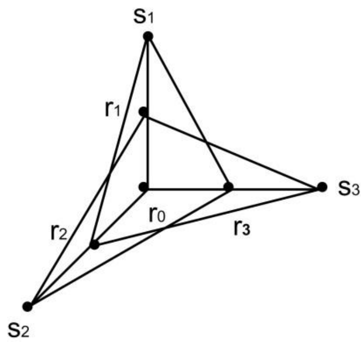

Figure 3. Graph $CD(St_{3,3,1})$

Since the double star graph $St_{2,2,1}$ is isomorphic to the path graph P_4, we have $CD(St_{2,2,1})$ is isomorphic to the discrete graph $\overline{K}_4$ (by Theorem 12, above).

Next we construct the circular distant divisor graph of $St_{n,n,1}$, when n is not a prime number.

Theorem 21. Let $St_{n,n,1}$ be the double star graph, where n is not a prime number. Then the circular distant divisor graph $CD(St_{n,n,1})$ is isomorphic to

$K_{2n+1},$ if n is a multiple of 12.

$St_{n,n,1} \cup \{r_0 s_i, r_i r_j, s_i s_j\},$ if n is a multiple of 4 but not a multiple of 12.

$St_{n,n,1} \cup \{r_i s_j\},$ if n is a multiple of 3 and $n>6.$

$St_{n,n,1} \cup \{r_0 s_i, r_i r_j\},$ if n is not a multiple of 3.

$St_{n,n,1},$ if n is product of two primes greater than 3.

Proof. In the double star graph, $St_{n,n,1}$, we observe that various circular distances are

$cir(r_0, r_i) = cir(r_i, s_i) = 2 \ \ \forall i.$

$cir(r_0, s_i) = cir(r_i, r_j) = 4 \ \ \forall i \neq j.$

$cir(r_i, s_j) = 6$ and $cir(s_i, s_j) = 8 \ \ \forall i \neq j.$

Now we check, whether the circular distances 2, 4,

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6, 8 are divisors of number of edges $2n$ or not.

When n is a multiple of 12, the above all circular distances are divisors of $2n.$ So there exist circular distant divisor paths from r_0 to r_i, r_i to s_i, r_0 to s_i, r_i to r_j, r_i to s_j and from s_i to $s_j.$ Thus there is an edge between each pair of vertices. Hence $CD(St_{n,n,1}) \cong K_{2n+1}$.

When n is a multiple of 4 but not multiple of 12, $cir(r_i, s_j) = 6$ does not divide $2n$ and the remaining circular distances are divisors of $2n.$ So there exist circular distant divisor paths from r_0 to r_i, r_i to s_i, r_0 to s_i, r_i to r_j and from s_i to $s_j.$ Thus there is an edge between any pair of these vertices. Hence $CD(St_{n,n,1}) \cong St_{n,n,1} \cup \{r_0s_i, r_ir_j, s_is_j\}.$

When n is a multiple of 3 and n is greater than 6, the distances $cir(r_0, s_i) = 4, cir(r_i, r_j) = 4, cir(s_i, s_j) = 8$ do not divide $2n$ and all the remaining distances are divisors of $2n.$ So there exists circular distant divisor paths from r_0 to r_i, r_i to s_i and from r_i to $s_j.$ Thus there is an edge between these vertices. Hence $CD(St_{n,n,1}) \cong St_{n,n,1} \cup \{r_is_j\}.$

For $n = 6,$ all circular distances are divisors of $12 = 2n,$ except $cir(s_i, s_j) = 8.$

Hence $CD(St_{6,6,1}) \cong St_{6,6,1} \cup \{r_0s_i, r_ir_j, r_is_j\}.$

When n is not a multiple of 3, $cir(r_i, s_j) = 6,$ $cir(s_i, s_j) = 8$ do not divide $2n$ and remaining all are divisors of $2n.$ So there exist circular distant divisor paths from r_0 to r_i, r_i to s_i, r_0 to s_i and from r_i to $r_j.$ Thus there exist an edge between these vertices. Hence $CD(St_{n,n,1}) \cong St_{n,n,1} \cup \{r_0s_i, r_ir_j\}.$

When n is a product of two primes and $n > 3,$ only $cir(r_0, s_i) = 2, cir(r_i, s_i) = 2$ are divisors of $2n$ and remaining are not divisors of $2n.$ So there exists circular distant divisor paths from r_0 to r_i and from r_i to s_i. Thus there exists an edge between these vertices. Hence $CD(St_{n,n,1}) \cong St_{n,n,1}.$

Corona graphs and circular distant divisor graphs

In this section, we construct the circular distant divisor graphs of some classes of corona graphs. We begin with the study of the structure of corona graph of complete graph

Theorem 27. Let K_n^+ be the corona graph of complete graph with $2n$ vertices. Then we have

$$CD(K_n^+) \cong \begin{cases} K_n^+ & \text{if } n(n+1) \text{ is a multiple of } 4 \text{ and } n \text{ is odd} \\ nK_2 & \text{if } n(n+1) \text{ is a multiple of } 4 \text{ and } n \text{ is even} \\ K_n + \overline{K}_n & \text{if } n(n+1) \text{ is not a multiple of } 4 \text{ and } n \text{ is odd} \\ \overline{K}_n & \text{if } n(n+1) \text{ is not a multiple of } 4, 2n, 2n+4, 2n+8 \\ K_n^+ & \text{if } n(n+1) \text{ is a multiple of } 4 \text{ and } 2n+8 \end{cases}$$

Proof. In the corona graph of complete graph, K_n^+, let $r_1, r_2, \cdots, r_n$ be the vertices of K_n and let $s_1, s_2, \cdots, s_n$ be the pendent vertices attached to $r_1, r_2, \cdots, r_n$ respectively. Then we can see that

$cir(r_i, r_j) = n,$ for all $1 \le i \ne j \le n.$

$cir(r_i, s_i) = 2,$ for all $1 \le i \le n.$

$cir(r_i, s_j) = n+2,$ for all $1 \le i \ne j \le n.$

$cir(s_i, s_j) = n+4,$ for all $1 \le i \ne j \le n.$

Observe that K_n^+ has $2n$ vertices and $\binom{n+1}{2}$ edges.

To construct the circular distant divisor graph, we need to check whether the circular distances 2, n, $n+2$ and $n+4$ are divisors of the number of edges or not.

Case 1 Suppose $n(n+1)$ is multiple of 4 and n is odd.

In this case, $n(n+1)$ is multiple of 4 implies $n(n+1)/2 \equiv 0 (mod\ 2)$ and n is odd implies $n(n+1)/2 \equiv 0 (mod\ n).$ So only 2 and n divides $n(n+1)/2.$ So there exists a circular distant divisor path from r_i to r_j for all $i, j, i \ne j$ and from r_i to s_i for all i in $CD(K_n^+).$ Hence $CD(K_n^+) \cong K_n^+$.

See Example 28, below.

Case 2 Suppose $n(n+1)$ is multiple of 4 and n is even

In this case, $n(n+1)$ is multiple of 4 implies

$n(n+1)/2 \equiv 0(mod\ 2)$. So only 2 divides $n(n+1)/2$. So there exists a circular distant divisor path from r_i to s_i for all i. Hence $CD(K_n^+) \cong nK_2$.
See Example 29, below.

Case 3 Suppose $n(n+1)$ is not multiple of 4 and n is odd.
In this case, n is odd implies $n(n+1)/2 \equiv 0(mod\ n)$. So only n divides $n(n+1)/2$. So there exists a circular distant divisor path from r_i to r_j and remaining all are isolated. Hence $CD(K_n^+) \cong K_n + \overline{K}_n$
See Example 30, below.

Case 4 Suppose $n(n+1)$ is not multiple of 4, $2n$, $2n+4$, $2n+8$.
In this case, clearly no circular distant divisor path exists between any pair of vertices. So in $CD(K_n^+)$, each vertex is isolated. Hence $CD(K_n^+) \cong \overline{K}_n$.
See Example 31, below.

Case 5 Suppose $n(n+1)$ is multiple of 4 and $2n+8$.
In this case, clearly $n(n+1)/2 \equiv 0(mod\ 2)$ and $n(n+1)/2 \equiv 0(mod\ n+4)$. Hence only 2 and $n+4$ are divisors of $n(n+1)/2$. So there exists circular distant divisor path from r_i to s_i and from s_i to s_j. Thus $CD(K_n^+) \cong K_n^+$. See Example 32, below.

Example 28. For K_3^+, $CD(K_3^+) \cong K_3^+$.

Example 29. For K_4^+, $CD(K_4^+) \cong 4K_2$.

Example 30. For K_5^+, $CD(K_5^+) \cong K_5 + \overline{K}_5$.

Example 31. For K_6^+, $CD(K_6^+) \cong \overline{K}_6$.

Example 32. For K_8^+, $CD(K_8^+) \cong K_8^+$

Next, look at circular distant divisor graph of corona graph of a cycle graph.

Theorem 33. Let C_n^+ be the corona graph of cycle graph with $2n\ (n \geq 5)$ vertices. Then we have $CD(C_n^+)$ is isomorphic to K_n^+. Further we have $CD(C_3^+) \cong K_3^+$ and $CD(C_4^+) \cong 2K_4 + 4K_2$.

Proof. In corona graph of cycle graph, C_n^+, let $r_1, r_2, \cdots, r_n$ be the vertices of C_n and let s_i be the pendent vertex attached to r_i for all $1 \leq i \leq n$.
Observe that C_n^+ has $2n$ vertices and $2n$ edges and furthermore, the various circular distances are

$cir(r_i, r_j) = n, \forall\ i \neq j.$
$cir(r_i, s_i) = 2\ \forall\ i.$
$cir(r_i, s_j) = n+2\ \forall\ i \neq j.$
$cir(s_i, s_j) = n+4\ \ \forall\ i \neq j,$ for $i, j = 1, 2, \cdots, n.$

To construct the circular distant divisor graph, we need to check the numbers 2, n, $n+2$ and $n+4$ are divisors of number of edges or not.

For $n=3$, number of edges in C_3^+ is 6. Now $cir(r_i, r_j) = 3$, $cir(r_i, s_i) = 2$. Hence $CD(C_3^+) \cong K_3^+$.

For $n=4$, number of edges in C_4^+ is 8. Now $cir(r_i, r_j) = 4$, $cir(r_i, s_i) = 2$ and $cir(s_i, s_j) = 8$. Hence $CD(C_4^+) \cong 2K_4 + 4K_2$.

Now we construct $CD(C_n^+)$, when $n \geq 5$.
Since the number of edges of $CD(C_n^+)$ is $2n$ and it is a multiple of 2 and n, there exists a circular distant divisor path from r_i to r_j and from r_i to s_i in $CD(C_n^+)$. Further $2n$ is not a multiple of $n+2$ and $n+4$. So there is no circular distant divisor path from r_i to s_j and from s_i to s_j. Hence $CD(C_n^+) \cong K_n^+$.

See below Example 34.

Example 34. For C_5^+, $CD(C_5^+) \cong K_5^+$.

Next, we consider the corona graph of a path graph.

Theorem 35. Let P_n^+ be the corona graph of path graph with $2n\ (n \geq 2)$ vertices. Then we have $CD(P_n^+) \cong \overline{K}_{2n}$

Proof. In corona graph of path graph, P_n^+, let $r_1, r_2, \cdots, r_n$ be the vertices of P_n and let s_i be the pendent vertex attached to r_i for all $1 \le i \le n$. Observe that P_n^+ has $2n$ vertices and $2n-1$ edges.

In P_n^+ we observe that

$cir(r_i, r_j) = 2|i-j| \forall i \ne j.$

$cir(r_i, s_i) = 2, \forall i.$

$cir(r_i, s_j) = 2 + 2|i-j| \ \forall i \ne j.$

$cir(s_i, s_j) = 4 + 2|i-j| \forall i \ne j,$ for $i, j = 1, 2, \cdots, n$.

Now the circular distance between any two vertices of P_n^+ is an even number, but number of edges is $2n-1$, which is not a multiple of any even number. So there is no circular distant divisor path between any two vertices. Hence $CD(P_n^+) \cong \overline{K}_{2n}$.

Next, we pay attention on corona graph of a star graph.

Theorem 36. Let $St_{n,1}^+$ be the corona graph of star graph with $2n+2$ vertices. Then we have $CD(St_{n,1}^+) \cong \overline{K}_{2n+2}$.

Proof. In corona graph of star graph, $St_{n,1}^+$, let $r_0, r_1, r_2, \cdots, r_n$ be the vertices of $St_{n,1}^+$ and let $s_0, s_1, s_2, \cdots, s_n$ be the pendent vertices attached to $r_0, r_1, r_2, \cdots, r_n$ respectively. Observe that $St_{n,1}^+$ has $2n+2$ vertices and $2n+1$ edges. Now , we have

$cir(r_0, r_i) = cir(r_i, s_i) = 2, \forall i$

$cir(r_i, r_j) = cir(s_0, r_i) = 4 \ \ \forall i \ne j.$

$cir(r_i, s_j) = cir(s_0, s_i) = 6 \ \ \forall i \ne j.$

$cir(s_i, s_j) = 8 \ \ \forall i \ne j,$ for $i, j = 1, 2, \cdots, n$.

Now the circular distance between any two vertices of $St_{n,1}^+$ is an even number, but number of edges $2n+1$, is not a multiple of any even number. So there does not exists any circular distant divisor path between any two vertices. Hence $CD(St_{n,1}^+) \cong \overline{K}_{2n+2}$.

Finally, we have

Theorem 37. Let $St_{n,n,1}^+$ be the corona graph of double star graph with $4n+2$ vertices. Then we have $CD(St_{n,n,1}^+) \cong \overline{K}_{4n+2}$.

Proof. In the corona graph of double star graph, $St_{n,n,1}^+$, let $\{r_0\} \cup \{r_1, r_2, \cdots, r_n\} \cup \{s_1, s_2, \cdots, s_n\}$ be the vertices of $St_{n,n,1}$ and let $l_0, l_1, l_2, \cdots, l_n$, $m_1, m_2, \cdots, m_n$ be the pendent vertices attached to $r_0, r_1, r_2, \cdots, r_n, s_1, s_2, \cdots, s_n$ respectively. In $St_{n,n,1}^+$, we observe that the circular distance between any two vertices is an even number. But the number of edges in $St_{n,n,1}^+$, is $4n+1$, which is odd. So there does not exists any circular distant divisor path between any two vertices. Hence $CD(St_{n,n,1}^+) \cong \overline{K}_{4n+2}$.

Conclusion

In this work, construction of circular distant divisor graphs have been constructed. We constructed circular distant divisor graphs of families of graphs and also constructed circular distant divisor graphs of corona graphs of some families of graphs.

In future work, we want to study construction of circular $D-$distant divisor graphs and establish the relation between circular distant divisor graphs and circular $D-$distant divisor graphs. Further we may study the properties of circular $D-$distant divisor graphs.

References

P. L. N Varma, P. L. N., & Veeranjaneyulu, J. Study of circular distance in graphs, Turkish Journal of Computer and Mathematics Education, 12(2), 2021; 2437-2444.

R. Aravamudhan and B. Rajendran, Graph equations involving antipodal Graphs, Presented at the Seminar on Combinatorics and applications. ISI, Calcutta, 1984, 40-43.

R. Aravamudhan and B. Rajendran, On antipodal Graphs, Discrete Math. 1984; 49, 193-195. DOI: 10.1016/0012-365X(84)90117-1.

J.A.Bonday and U.S.R.Murthy, Graph theory with Applications, Springer London, 2008.

F. Buckley and F. Harary, Distance in graphs, Addison-Wesley, Redwood City, 1990.

David M. Burton, Elementary Number Theory, Tata McGraw-Hill Publishing Co. Ltd, 2006.

S. Saravana Kumar and K. Nagarajan, Distant Divisor Graphs, International J. of Math. Sci & Engg. Appls. 2013; 7, 83-97.

S. Saravana Kumar and K. Nagarajan, Distant Divisor Graph of Subdivision of a Graph, International Journal of Mathematics and Soft Computing. 2014; 4, 51-60.

S. Saravana Kumar and K. Nagarajan, Detour Distant Divisor Graphs, International Journal of Mathematics Trends and Technology (IJMTT). 2014; 9, 1-15.

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