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Fixed Point Theorems of Almost Generalized Contractive Mappings in b -Metric Spaces and an Application to Integral Equation

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Abstract: In this study, we have new fixed point results for weak contraction mappings in complete and partially ordered b -metric spaces. Our findings expand and generalize the results of Jachymski and Mitoku et al and many more results in the literature as well. To illustrate our work, we present an application on the existence and uniqueness of a nonlinear quadratic integral problem solution. Moreover, an open problem is presented to enable the scope for future research in this area.

Keywords: b -metric space; weak contractions; fixed point, compatible and weakly compatible mappings; coupled coincidence point; nonlinear quadratic integral equations

MSC: 47H10; 54H25



Citation: Seshagiri Rao, N.; Mitrović, Z.D.; Santana, D.; Mlaiki, N. Fixed Point Theorems of Almost Generalized Contractive Mappings in b -Metric Spaces and an Application to Integral Equation. *Mathematics* **2023**, *11*, 2580. <https://doi.org/10.3390/math11112580>

Academic Editors: Talat Nazir and Óscar Valero Sierra

Received: 17 March 2023

Revised: 27 May 2023

Accepted: 2 June 2023

Published: 5 June 2023



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1. Introduction and Preliminaries

The concept of b -metric spaces is considered the most important generalization to the metric spaces. Recently, fixed points of contractive mappings in b -metric spaces have been applied in pure and applied mathematics, with several applications for scientific problems. Fixed points results in b -metric spaces are very useful to many scholars. This concept was first introduced by Bakhtin [1] in 1983, and later was expanded by Czerwik [2]. In 2004, Ran and Reurings [3] initiated fixed point results in partially ordered b -metric space. Since then, the idea has been generalized and extended by many authors in many different metric spaces, with contraction conditions found in sources such as [4–26]. Additionally, these results have been applied to differential equations, including differential and integral equations, to find unique solutions.

First of all, we remind the reader of the definition of partially ordered b -metric spaces.

Definition 1 ([6]). A mapping $\rho : \mathfrak{S} \times \mathfrak{S} \rightarrow [0, +\infty)$, where \mathfrak{S} is a non-empty set is known to be a b -metric, if ρ satisfies the below properties for any $\theta_1, \theta_2, \theta_3 \in \mathfrak{S}$ and for some real number $s \geq 1$,

- $\rho(\theta_1, \theta_2) = 0$ if and only if $\theta_1 = \theta_2$;
- $\rho(\theta_1, \theta_2) = \rho(\theta_2, \theta_1)$;
- $\rho(\theta_1, \theta_2) \leq s(\rho(\theta_1, \theta_3) + \rho(\theta_3, \theta_2))$.

Then (\mathfrak{S}, ρ, s) is known as a b -metric space. If (\mathfrak{S}, \preceq) is still a partially ordered set, then $(\mathfrak{S}, \rho, s, \preceq)$ is called a partially ordered b -metric space.

Definition 2 ([6]). Let $(\mathfrak{S}, \rho, s, \preceq)$ be a b -metric space. Then

- (1). a sequence $\{\theta_n\}$ is said to converge to θ , if $\rho(\theta_n, \theta) \rightarrow 0$ as $n \rightarrow +\infty$ and written as $\lim_{n \rightarrow +\infty} \theta_n = \theta$.
- (2). $\{\theta_n\}$ is said to be a Cauchy sequence in \mathfrak{S} , if $\rho(\theta_n, \theta_m) \rightarrow 0$, as $n, m \rightarrow +\infty$.
- (3). $(\mathfrak{S}, \rho, \preceq)$ is said to be complete, if every Cauchy sequence in it is convergent.

Definition 3. If the metric ρ is complete, then $(\mathfrak{S}, \rho, \preceq)$ is called a complete partially ordered b-metric space (c.p.o.b-m.s.).

Definition 4 ([6]). Let (\mathfrak{S}, \preceq) be a partially ordered set and let $\ell, \mathcal{B} : \mathfrak{S} \rightarrow \mathfrak{S}$ be two mappings. Then:

- (1) \mathcal{B} is called a monotone non-decreasing, if $\mathcal{B}(\theta) \preceq \mathcal{B}(\xi)$ for all $\theta, \xi \in \mathfrak{S}$ with $\theta \preceq \xi$;
- (2) An element $\theta \in \mathfrak{S}$ is called a coincidence (common fixed) point of ℓ and \mathcal{B} , if $\ell\theta = \mathcal{B}\theta$ ($\ell\theta = \mathcal{B}\theta = \theta$);
- (3) ℓ and \mathcal{B} are called commuting, if $\ell\mathcal{B}\theta = \mathcal{B}\ell\theta$, for all $\theta \in \mathfrak{S}$;
- (4) ℓ and \mathcal{B} are called compatible, if any sequence $\{\theta_n\}$ with $\lim_{n \rightarrow +\infty} \ell\theta_n = \lim_{n \rightarrow +\infty} \mathcal{B}\theta_n = \mu$, for $\mu \in \mathfrak{S}$ then $\lim_{n \rightarrow +\infty} \rho(\mathcal{B}\ell\theta_n, \ell\mathcal{B}\theta_n) = 0$;
- (5) A pair of self maps (ℓ, \mathcal{B}) is called weakly compatible, if $\ell\mathcal{B}\theta = \mathcal{B}\ell\theta$, when $\mathcal{B}\theta = \ell\theta$ for some $\theta \in \mathfrak{S}$;
- (6) \mathcal{B} is called a monotone ℓ -non-decreasing, if

$$\ell\theta \preceq \ell\xi \text{ implies } \mathcal{B}\theta \preceq \mathcal{B}\xi, \text{ for any } \theta, \xi \in \mathfrak{S};$$

- (7) A non-empty set \mathfrak{S} is called well ordered set, if very two elements of it are comparable, i.e., $\theta \preceq \xi$ or $\xi \preceq \theta$, for $\theta, \xi \in \mathfrak{S}$.

Definition 5 ([6]). Let $(\mathfrak{S}, \rho, \preceq)$ be a partially ordered set, and let $\mathcal{B} : \mathfrak{S} \times \mathfrak{S} \rightarrow \mathfrak{S}$ and $\ell : \mathfrak{S} \rightarrow \mathfrak{S}$ be two mappings. Then:

- (1) \mathcal{B} has the mixed ℓ -monotone property, if \mathcal{B} is a non-decreasing ℓ -monotone in its first argument and is a non-increasing ℓ -monotone in its second argument, that is for any $\theta, \xi \in \mathfrak{S}$

$$\begin{aligned} \theta_1, \theta_2 \in \mathfrak{S}, \ell\theta_1 \preceq \ell\theta_2 \text{ implies } \mathcal{B}(\theta_1, \xi) \preceq \mathcal{B}(\theta_2, \xi), \text{ and} \\ \xi_1, \xi_2 \in X, \ell\xi_1 \preceq \ell\xi_2 \text{ implies } \mathcal{B}(\theta, \xi_1) \succeq \mathcal{B}(\theta, \xi_2). \end{aligned}$$

Suppose, if ℓ is an identity mapping then \mathcal{B} is said to have the mixed monotone property.

- (2) An element $(\theta, \xi) \in \mathfrak{S} \times \mathfrak{S}$ is called a coupled coincidence point of \mathcal{B} and ℓ , if $\mathcal{B}(\theta, \xi) = \ell\theta$ and $\mathcal{B}(\xi, \theta) = \ell\xi$. Note that if ℓ is an identity mapping, then (θ, ξ) is said to be a coupled fixed point of \mathcal{B} .
- (3) Element $\theta \in \mathfrak{S}$ is called a common fixed point of \mathcal{B} and ℓ , if $\mathcal{B}(\theta, \theta) = \ell\theta = \theta$.
- (4) \mathcal{B} and ℓ are commutative, if for all $\theta, \xi \in \mathfrak{S}$, $\mathcal{B}(\ell\theta, \ell\xi) = \ell(\mathcal{B}\theta, \mathcal{B}\xi)$.
- (5) \mathcal{B} and ℓ are said to be compatible, if

$$\lim_{n \rightarrow +\infty} \rho(\ell(\mathcal{B}(\theta_n, \xi_n)), \mathcal{B}(\ell\theta_n, \ell\xi_n)) = 0$$

and

$$\lim_{n \rightarrow +\infty} \rho(\ell(\mathcal{B}(\xi_n, \theta_n)), \mathcal{B}(\ell\xi_n, \ell\theta_n)) = 0,$$

whenever $\{\theta_n\}$ and $\{\xi_n\}$ are any two sequences in \mathfrak{S} such that

$$\lim_{n \rightarrow +\infty} \mathcal{B}(\theta_n, \xi_n) = \lim_{n \rightarrow +\infty} \ell\theta_n = \theta$$

and

$$\lim_{n \rightarrow +\infty} \mathcal{B}(\xi_n, \theta_n) = \lim_{n \rightarrow +\infty} \ell\xi_n = \xi,$$

for all $\theta, \xi \in \mathfrak{S}$.

The following lemma is very useful in proving the convergence of a sequence in b -metric spaces.

Lemma 1 ([6]). *Let $(\mathfrak{S}, \rho, s, \preceq)$ be a b -metric space with $s > 1$, and suppose that $\{\theta_n\}$ and $\{\xi_n\}$ are b -convergent to θ and ξ , respectively. Then*

$$\frac{1}{s^2} \rho(\theta, \xi) \leq \liminf_{n \rightarrow +\infty} \rho(\theta_n, \xi_n) \leq \limsup_{n \rightarrow +\infty} \rho(\theta_n, \xi_n) \leq s^2 \rho(\theta, \xi).$$

In particular, if $\theta = \xi$, then $\lim_{n \rightarrow +\infty} \rho(\theta_n, \xi_n) = 0$. Moreover, for each $\tau \in X$, we have

$$\frac{1}{s} \rho(\theta, \tau) \leq \liminf_{n \rightarrow +\infty} \rho(\theta_n, \tau) \leq \limsup_{n \rightarrow +\infty} \rho(\theta_n, \tau) \leq s \rho(\theta, \tau).$$

Throughout the rest of this manuscript we use the following altering distance functions.

- (i) $\tilde{\Phi} = \{\tilde{\phi} : \tilde{\phi}$ is a continuous, non-decreasing self mapping on $[0, +\infty)$ such that $\tilde{\phi}(\varepsilon) = 0$ if and only if $\varepsilon = 0\}$.
- (ii) $\tilde{\Psi} = \{\tilde{\psi} : \tilde{\psi}$ is a lower semi-continuous self mapping on $[0, +\infty)$ such that $\tilde{\psi}(\varepsilon) = 0$ if and only if $\varepsilon = 0\}$.
- (iii) $\Theta = \{\Omega : \Omega$ is a self mapping on $[0, +\infty)$ such that $\Omega(\varepsilon) = 0$ if and only if $\varepsilon = 0\}$.

Next, we introduce the concept of generalized weak contraction involving the altering distance functions $\tilde{\phi} \in \tilde{\Phi}$, $\tilde{\psi} \in \tilde{\Psi}$ and $\Omega \in \Theta$ for a self mapping \mathcal{B} on \mathfrak{S} in a c.p.o. b -m.s.

$$\tilde{\phi}(s\rho(\mathcal{B}\theta, \mathcal{B}\zeta)) \leq \tilde{\phi}(\mathcal{E}(\theta, \zeta)) - \tilde{\psi}(\mathcal{E}(\theta, \zeta)) + \Lambda\Omega(\mathcal{F}(\theta, \zeta)), \tag{1}$$

for any $\theta, \zeta \in \mathfrak{S}$ with $\theta \preceq \zeta$ and $\Lambda \geq 0$, where

$$\mathcal{E}(\theta, \zeta) = \max\left\{ \frac{\rho(\zeta, \mathcal{B}\zeta)[1 + \rho(\theta, \mathcal{B}\theta)]}{1 + \rho(\theta, \zeta)}, \frac{\rho(\theta, \mathcal{B}\theta) \rho(\zeta, \mathcal{B}\zeta)}{1 + \rho(\theta, \zeta)}, \frac{\rho(\theta, \mathcal{B}\zeta) + \rho(\zeta, \mathcal{B}\theta)}{2s}, \rho(\theta, \zeta) \right\}, \tag{2}$$

and

$$\mathcal{F}(\theta, \zeta) = \min\{\rho(\theta, \mathcal{B}\theta), \rho(\zeta, \mathcal{B}\zeta), \rho(\zeta, \mathcal{B}\theta), \rho(\theta, \mathcal{B}\zeta)\}. \tag{3}$$

The results obtained in this work generalize and extend the results in [4,5] and several comparable results in the literature. Furthermore, some variations of the results of [16,21,22,25,26] can be seen in this paper. We refer the reader to [6,17,24] for the basic definitions and the results which are necessary for understanding the present work.

2. Main Results

Now, we formulate and prove the theorem for the existence of a fixed point of the generalized weak contraction involving the altering distance functions in a c.p.o. b -m.s.

Theorem 1. *Let $(\mathfrak{S}, \rho, s, \preceq)$ be c.p.o. b -m.s. with $s > 1$, and \mathcal{B} is a continuous and non-decreasing self mapping on \mathfrak{S} such that it satisfies condition (1). If there exists $\theta_0 \in \mathfrak{S}$ such that $\theta_0 \preceq \mathcal{B}\theta_0$, then \mathcal{B} has a fixed point in \mathfrak{S} .*

Proof. If $\mathcal{B}\theta_0 = \theta_0$, for $\theta_0 \in \mathfrak{S}$ then the result is proved. Otherwise, $\theta_0 \prec \mathcal{B}\theta_0$ so then construct a sequence $\{\theta_n\}$ by $\theta_{n+1} = \mathcal{B}\theta_n$, for all $n \in \mathbb{N}$. As \mathcal{B} is an increasing mapping, then

$$\theta_0 \prec \mathcal{B}\theta_0 = \theta_1 \preceq \dots \preceq \theta_n \preceq \mathcal{B}\theta_n = \theta_{n+1} \preceq \dots \tag{4}$$

From (4), the result is also trivial, if $\theta_{n_0} = \theta_{n_0+1}$ for certain $n_0 \in \mathbb{N}$. Suppose not for all $n \geq 1$ then $\theta_n \succ \theta_{n-1}, (n \geq 1)$ and from condition (1), we have

$$\begin{aligned} \tilde{\phi}(\mathcal{P}(\theta_n, \theta_{n+1})) &= \tilde{\phi}(\mathcal{P}(\mathcal{B}\theta_{n-1}, \mathcal{B}\theta_n)) \leq \tilde{\phi}(\mathcal{J}\mathcal{P}(\mathcal{B}\theta_{n-1}, \mathcal{B}\theta_n)) \\ &\leq \tilde{\phi}(\mathcal{E}(\theta_{n-1}, \theta_n)) - \tilde{\psi}(\mathcal{E}(\theta_{n-1}, \theta_n)) + \Lambda\Omega(\mathcal{F}(\theta_n, \theta_{n+1})), \end{aligned} \tag{5}$$

where

$$\begin{aligned} \mathcal{E}(\theta_{n-1}, \theta_n) &= \max\left\{ \frac{\mathcal{P}(\theta_n, \mathcal{B}\theta_n)[1 + \mathcal{P}(\theta_{n-1}, \mathcal{B}\theta_{n-1})]}{1 + \mathcal{P}(\theta_{n-1}, \theta_n)}, \frac{\mathcal{P}(\theta_{n-1}, \mathcal{B}\theta_{n-1}) \mathcal{P}(\theta_n, \mathcal{B}\theta_n)}{1 + \mathcal{P}(\theta_{n-1}, \theta_n)}, \right. \\ &\quad \left. \frac{\mathcal{P}(\theta_{n-1}, \mathcal{B}\theta_n) + \mathcal{P}(\theta_n, \mathcal{B}\theta_{n-1})}{2\mathcal{J}}, \mathcal{P}(\theta_{n-1}, \theta_n) \right\} \\ &\leq \max\left\{ \mathcal{P}(\theta_n, \theta_{n+1}), \frac{\mathcal{P}(\theta_{n-1}, \theta_n) + \mathcal{P}(\theta_n, \theta_{n+1})}{2}, \mathcal{P}(\theta_{n-1}, \theta_n) \right\} \\ &\leq \max\{ \mathcal{P}(\theta_n, \theta_{n+1}), \mathcal{P}(\theta_{n-1}, \theta_n) \} \end{aligned} \tag{6}$$

and

$$\begin{aligned} \mathcal{F}(\theta_{n-1}, \theta_n) &= \min\{ \mathcal{P}(\theta_{n-1}, \mathcal{B}\theta_{n-1}), \mathcal{P}(\theta_n, \mathcal{B}\theta_n), \mathcal{P}(\theta_n, \mathcal{B}\theta_{n-1}), \mathcal{P}(\theta_{n-1}, \mathcal{B}\theta_n) \} \\ &= \min\{ \mathcal{P}(\theta_{n-1}, \theta_n), \mathcal{P}(\theta_n, \theta_{n+1}), \mathcal{P}(\theta_n, \theta_n), \mathcal{P}(\theta_{n-1}, \theta_{n+1}) \} \\ &= 0. \end{aligned} \tag{7}$$

Therefore from Equations (5)–(7), we obtained that

$$\mathcal{P}(\theta_n, \theta_{n+1}) = \mathcal{P}(\mathcal{B}\theta_{n-1}, \mathcal{B}\theta_n) \leq \frac{1}{\mathcal{J}} \mathcal{E}(\theta_{n-1}, \theta_n). \tag{8}$$

Assume that for some $n \geq 1$, $\max\{ \mathcal{P}(\theta_n, \theta_{n+1}), \mathcal{P}(\theta_{n-1}, \theta_n) \} = \mathcal{P}(\theta_n, \theta_{n+1})$, then from Equation (8) we obtain

$$\mathcal{P}(\theta_n, \theta_{n+1}) \leq \frac{1}{\mathcal{J}} \mathcal{P}(\theta_n, \theta_{n+1}), \tag{9}$$

which is a contradiction. So, $\max\{ \mathcal{P}(\theta_n, \theta_{n+1}), \mathcal{P}(\theta_{n-1}, \theta_n) \} = \mathcal{P}(\theta_{n-1}, \theta_n), (n \geq 1)$. Thus, from (8), we have

$$\mathcal{P}(\theta_n, \theta_{n+1}) \leq \frac{1}{\mathcal{J}} \mathcal{P}(\theta_{n-1}, \theta_n), \tag{10}$$

where $0 < \frac{1}{\mathcal{J}} < 1$. By the results of [12], we conclude that $\{\theta_n\}$ is a Cauchy sequence in \mathfrak{S} . Therefore, $\theta_n \rightarrow a \in \mathfrak{S}$ for some $a \in \mathfrak{S}$ by completeness of \mathfrak{S} .

Moreover, since \mathcal{B} is continuous, we obtain

$$\mathcal{B}a = \mathcal{B}\left(\lim_{n \rightarrow +\infty} \theta_n\right) = \lim_{n \rightarrow +\infty} \mathcal{B}\theta_n = \lim_{n \rightarrow +\infty} \theta_{n+1} = a. \tag{11}$$

So, $a \in \mathfrak{S}$ is a fixed point of \mathcal{B} . \square

Now we have the following result, assuming some condition on a space \mathfrak{S} .

Theorem 2. *If in Theorem 1 we replace the assumption about the continuity of the mapping \mathcal{B} with the following condition:*

$$\begin{aligned} &a \text{ sequence } \{\theta_n\} \text{ in } \mathfrak{S} \text{ is non-decreasing with } \theta_n \rightarrow a \in \mathfrak{S} \text{ then } \theta_n \preceq a, (n \geq 0), \\ &\text{that is, } a = \sup \theta_n, \end{aligned} \tag{12}$$

then the mapping \mathcal{B} has a fixed point in \mathfrak{S} .

Proof. As in proof of Theorem 1, we conclude that there exists a non-decreasing Cauchy sequence $\{\theta_n\} \subseteq \mathfrak{S}$ such that $\theta_n \rightarrow a \in \mathfrak{S}$. By condition (12), we obtain that $\theta_n \preceq a$, for all n , i.e., $a = \sup \theta_n$.

Next, to show that \mathcal{B} has a fixed point a , let $\mathcal{B}a \neq a$, then

$$\mathcal{E}(\theta_n, a) = \max\left\{\frac{\rho(a, \mathcal{B}a)[1 + \rho(\theta_n, \mathcal{B}\theta_n)]}{1 + \rho(\theta_n, a)}, \frac{\rho(\theta_n, \mathcal{B}\theta_n) \rho(a, \mathcal{B}a)}{1 + \rho(\theta_n, a)}, \frac{\rho(\theta_n, \mathcal{B}a) + \rho(a, \mathcal{B}\theta_n)}{2s}, \rho(\theta_n, a)\right\} \tag{13}$$

and

$$\mathcal{F}(\theta_n, a) = \min\{\rho(\theta_n, \mathcal{B}\theta_n), \rho(a, \mathcal{B}a), \rho(a, \mathcal{B}\theta_n), \rho(\theta_n, \mathcal{B}a)\}. \tag{14}$$

In Equations (13) and (14) by taking $n \rightarrow +\infty$, we obtain that

$$\lim_{n \rightarrow +\infty} \mathcal{E}(\theta_n, a) = \max\{\rho(a, \mathcal{B}a), 0, \frac{\rho(a, \mathcal{B}a)}{2s}, 0\} = \rho(a, \mathcal{B}a) \tag{15}$$

and

$$\lim_{n \rightarrow +\infty} \mathcal{F}(\theta_n, a) = \min\{0, \rho(a, \mathcal{B}a)\} = 0. \tag{16}$$

As $\theta_n \leq a, (n \geq 0)$, then from (1) we have

$$\begin{aligned} \tilde{\phi}(\rho(\theta_{n+1}, \mathcal{B}a)) &= \tilde{\phi}(\rho(\mathcal{B}\theta_n, \mathcal{B}a)) \\ &\leq \tilde{\phi}(s\rho(\mathcal{B}\theta_n, \mathcal{B}a) \leq \tilde{\phi}(\mathcal{E}(\theta_n, a)) - \tilde{\psi}(\mathcal{E}(\theta_n, a)) + \Lambda\Omega(\mathcal{F}(\theta_n, a)). \end{aligned} \tag{17}$$

By letting $n \rightarrow +\infty$ in (17), we obtain

$$\tilde{\phi}(\rho(a, \mathcal{B}a)) \leq \tilde{\phi}(\rho(a, \mathcal{B}a)) - \tilde{\psi}(\rho(a, \mathcal{B}a)) < \tilde{\phi}(\rho(a, \mathcal{B}a)), \tag{18}$$

which is a contradiction in (18). Hence, $\mathcal{B}a = a$. \square

Theorem 3. *The mapping \mathcal{B} in Theorems 1 and 2 has a unique fixed point, if every two elements of \mathfrak{S} are comparable.*

Proof. Assume that $\theta^*, \zeta^* \in \mathfrak{S}$ are the fixed points of \mathcal{B} with $\theta^* \neq \zeta^*$, then from Equation (1) we have

$$\begin{aligned} \tilde{\phi}(\rho(\mathcal{B}\theta^*, \mathcal{B}\zeta^*)) &\leq \tilde{\phi}(s\rho(\mathcal{B}\theta^*, \mathcal{B}\zeta^*)) \\ &\leq \tilde{\phi}(\mathcal{E}(\theta^*, \zeta^*)) - \tilde{\psi}(\mathcal{E}(\theta^*, \zeta^*)) + \Lambda\Omega(\mathcal{F}(\theta^*, \zeta^*)), \end{aligned} \tag{19}$$

where

$$\begin{aligned} \mathcal{E}(\theta^*, \zeta^*) &= \max\left\{\frac{\rho(\zeta^*, \mathcal{B}\zeta^*)[1 + \rho(\theta^*, \mathcal{B}\theta^*)]}{1 + \rho(\theta^*, \zeta^*)}, \frac{\rho(\theta^*, \mathcal{B}\theta^*) \rho(\zeta^*, \mathcal{B}\zeta^*)}{1 + \rho(\theta^*, \zeta^*)}, \frac{\rho(\theta^*, \mathcal{B}\zeta^*) + \rho(\zeta^*, \mathcal{B}\theta^*)}{2s}, \rho(\theta^*, \zeta^*)\right\} \\ &= \max\left\{\frac{\rho(\zeta^*, \zeta^*)[1 + \rho(\theta^*, \theta^*)]}{1 + \rho(\theta^*, \zeta^*)}, \frac{\rho(\theta^*, \theta^*) \rho(\zeta^*, \zeta^*)}{1 + \rho(\theta^*, \zeta^*)}, \frac{\rho(\theta^*, \zeta^*) + \rho(\zeta^*, \theta^*)}{2s}, \rho(\theta^*, \zeta^*)\right\} \\ &= \max\{0, 0, \frac{\rho(\theta^*, \zeta^*)}{s}, \rho(\theta^*, \zeta^*)\} \\ &= \rho(\theta^*, \zeta^*) \end{aligned} \tag{20}$$

and

$$\mathcal{F}(\theta^*, \zeta^*) = \min\{\rho(\theta^*, \mathcal{B}\theta^*), \rho(\zeta^*, \mathcal{B}\zeta^*), \rho(\zeta^*, \mathcal{B}\theta^*), \rho(\theta^*, \mathcal{B}\zeta^*)\} = 0. \tag{21}$$

From (19), we obtain

$$\rho(\theta^*, \zeta^*) = \rho(\mathcal{B}\theta^*, \mathcal{B}\zeta^*) \leq \frac{1}{j} \mathcal{E}(\theta^*, \zeta^*) < \mathcal{E}(\theta^*, \zeta^*) = \rho(\theta^*, \zeta^*), \tag{22}$$

which is a contradiction to $\theta^* \neq \zeta^*$. Thus, $\theta^* = \zeta^*$. Hence, the mapping \mathcal{B} has a unique fixed point in \mathfrak{S} . \square

Corollary 1. *The same conclusions will be achieved as from Theorems 1–3 by letting $\Lambda = 0$ in condition (1).*

Corollary 2. *In Corollary 1, by replacing $\tilde{\phi}(n) = n$ and $\tilde{\psi}(n) = (1 - k)n$, then one can obtain the same conclusions as in Theorems 1–3 with the following reduced contraction condition*

$$\rho(\mathcal{B}\theta, \mathcal{B}\zeta) \leq \frac{k}{j} \max\left\{ \frac{\rho(\zeta, \mathcal{B}\zeta)[1 + \rho(\theta, \mathcal{B}\theta)]}{1 + \rho(\theta, \zeta)}, \frac{\rho(\theta, \mathcal{B}\theta) \rho(\zeta, \mathcal{B}\zeta)}{1 + \rho(\theta, \zeta)}, \frac{\rho(\theta, \mathcal{B}\zeta) + \rho(\zeta, \mathcal{B}\theta)}{2j}, \rho(\theta, \zeta) \right\}. \tag{23}$$

Definition 6. *A self mapping \mathcal{B} over \mathfrak{S} is a generalized contraction with respect to a mapping $\ell : \mathfrak{S} \rightarrow \mathfrak{S}$, if it satisfies the following condition:*

$$\tilde{\phi}(j\rho(\mathcal{B}\theta, \mathcal{B}\zeta)) \leq \tilde{\phi}(\mathcal{E}_\ell(\theta, \zeta)) - \tilde{\psi}(\mathcal{F}_\ell(\theta, \zeta)) + \Lambda\Omega(\mathcal{F}_\ell(\theta, \zeta)), \tag{24}$$

where

$$\mathcal{E}_\ell(\theta, \zeta) = \max\left\{ \frac{\rho(\ell\zeta, \mathcal{B}\zeta)[1 + \rho(\ell\theta, \mathcal{B}\theta)]}{1 + \rho(\ell\theta, \ell\zeta)}, \frac{\rho(\ell\theta, \mathcal{B}\theta) \rho(\ell\zeta, \mathcal{B}\zeta)}{1 + \rho(\ell\theta, \ell\zeta)}, \frac{\rho(\ell\theta, \mathcal{B}\zeta) + \rho(\ell\zeta, \mathcal{B}\theta)}{2j}, \rho(\ell\theta, \ell\zeta) \right\} \tag{25}$$

and

$$\mathcal{F}_\ell(\theta, \zeta) = \min\{\rho(\ell\theta, \mathcal{B}\theta), \rho(\ell\zeta, \mathcal{B}\zeta), \rho(\ell\zeta, \mathcal{B}\theta), \rho(\ell\theta, \mathcal{B}\zeta)\}, \tag{26}$$

for all $\theta, \zeta \in \mathfrak{S}$ with $\ell\theta \preceq \ell\zeta$ and $\tilde{\phi} \in \tilde{\Phi}, \tilde{\psi} \in \tilde{\Psi}$ and $\Omega \in \Theta$.

Theorem 4. *Suppose that $(\mathfrak{S}, \rho, j, \preceq)$ is a c.p.o.b-m.s. Let \mathcal{B} and ℓ be continuous self mappings defined over \mathfrak{S} . If the mappings \mathcal{B} and ℓ satisfies the condition (24) such that*

- (i) \mathcal{B} is a monotone ℓ -non-decreasing;
- (ii) $\mathcal{B}\mathfrak{S} \subseteq \ell\mathfrak{S}$ and \mathcal{B}, ℓ are compatible;
- (iii) $\ell\theta_0 \preceq \mathcal{B}\theta_0$ for certain $\theta_0 \in \mathfrak{S}$;

then \mathcal{B} and ℓ have a coincidence point in \mathfrak{S} .

Proof. There exist two sequences $\{\theta_n\}$ and $\{\zeta_n\}$ in \mathfrak{S} by Theorem 2.2 of [14] such that

$$\zeta_n = \mathcal{B}\theta_n = \ell\theta_{n+1}, \forall n \geq 0, \tag{27}$$

for which

$$\ell\theta_0 \preceq \ell\theta_1 \preceq \dots \preceq \ell\theta_n \preceq \ell\theta_{n+1} \preceq \dots \tag{28}$$

From [14], we have to claim that

$$\rho(\zeta_n, \zeta_{n+1}) \leq \lambda\rho(\zeta_{n-1}, \zeta_n), (n \geq 1), \tag{29}$$

where $\lambda = \frac{1}{j}, j > 1$. Therefore, from Equations (24), (27) and (28), we obtain that

$$\begin{aligned} \tilde{\phi}(j\rho(\zeta_n, \zeta_{n+1})) &= \tilde{\phi}(j\rho(\mathcal{B}\theta_n, \mathcal{B}\theta_{n+1})) \\ &\leq \tilde{\phi}(\mathcal{E}_\ell(\theta_n, \theta_{n+1})) - \tilde{\psi}(\mathcal{F}_\ell(\theta_n, \theta_{n+1})) + \Lambda\Omega(\mathcal{F}_\ell(\theta_n, \theta_{n+1})), \end{aligned} \tag{30}$$

where

$$\begin{aligned}
 \mathcal{E}_\ell(\theta_n, \theta_{n+1}) &= \max\left\{ \frac{\mathcal{P}(\ell\theta_{n+1}, \mathcal{B}\theta_{n+1})[1 + \mathcal{P}(\ell\theta_n, \mathcal{B}\theta_n)]}{1 + \mathcal{P}(\ell\theta_n, \ell\theta_{n+1})}, \right. \\
 &\quad \left. \frac{\mathcal{P}(\ell\theta_n, \mathcal{B}\theta_n) \mathcal{P}(\ell\theta_{n+1}, \mathcal{B}\theta_{n+1})}{1 + \mathcal{P}(\ell\theta_n, \ell\theta_{n+1})}, \frac{\mathcal{P}(\ell\theta_n, \mathcal{B}\theta_{n+1}) + \mathcal{P}(\ell\theta_{n+1}, \mathcal{B}\theta_n)}{2\mathcal{J}}, \right. \\
 &\quad \left. \mathcal{P}(\ell\theta_n, \ell\theta_{n+1}) \right\} \\
 &= \max\left\{ \frac{\mathcal{P}(\zeta_n, \zeta_{n+1})[1 + \mathcal{P}(\zeta_{n-1}, \zeta_n)]}{1 + \mathcal{P}(\zeta_{n-1}, \zeta_n)}, \frac{\mathcal{P}(\zeta_{n-1}, \zeta_n) \mathcal{P}(\zeta_n, \zeta_{n+1})}{1 + \mathcal{P}(\zeta_{n-1}, \zeta_n)}, \right. \\
 &\quad \left. \frac{\mathcal{P}(\zeta_{n-1}, \zeta_{n+1}) + \mathcal{P}(\zeta_n, \zeta_n)}{2\mathcal{J}}, \mathcal{P}(\zeta_{n-1}, \zeta_n) \right\} \\
 &\leq \max\left\{ \mathcal{P}(\zeta_n, \zeta_{n+1}), \frac{\mathcal{P}(\zeta_{n-1}, \zeta_n) + \mathcal{P}(\zeta_n, \zeta_{n+1})}{2}, \mathcal{P}(\zeta_{n-1}, \zeta_n) \right\} \\
 &\leq \max\left\{ \mathcal{P}(\zeta_{n-1}, \zeta_n), \mathcal{P}(\zeta_n, \zeta_{n+1}) \right\}
 \end{aligned} \tag{31}$$

and

$$\begin{aligned}
 \mathcal{F}_\ell(\theta_n, \theta_{n+1}) &= \min\{\mathcal{P}(\ell\theta_n, \mathcal{B}\theta_n), \mathcal{P}(\ell\theta_{n+1}, \mathcal{B}\theta_{n+1}), \mathcal{P}(\ell\theta_{n+1}, \mathcal{B}\theta_n), \mathcal{P}(\ell\theta_n, \mathcal{B}\theta_{n+1})\} \\
 &= \min\{\mathcal{P}(\zeta_{n-1}, \zeta_n), \mathcal{P}(\zeta_n, \zeta_{n+1}), \mathcal{P}(\zeta_n, \zeta_n), \mathcal{P}(\zeta_{n-1}, \zeta_{n+1})\} \\
 &= 0.
 \end{aligned} \tag{32}$$

Thus, from Equations (30)–(32), it follows that

$$\begin{aligned}
 \tilde{\Phi}(\mathcal{J}\mathcal{P}(\zeta_n, \zeta_{n+1})) &\leq \tilde{\Phi}(\max\{\mathcal{P}(\zeta_{n-1}, \zeta_n), \mathcal{P}(\zeta_n, \zeta_{n+1})\}) \\
 &\quad - \tilde{\Psi}(\max\{\mathcal{P}(\zeta_{n-1}, \zeta_n), \mathcal{P}(\zeta_n, \zeta_{n+1})\}).
 \end{aligned} \tag{33}$$

Suppose $0 < \mathcal{P}(\zeta_{n-1}, \zeta_n) \leq \mathcal{P}(\zeta_n, \zeta_{n+1})$ for some n , then (33) implies that

$$\tilde{\Phi}(\mathcal{J}\mathcal{P}(\zeta_n, \zeta_{n+1})) \leq \tilde{\Phi}(\mathcal{P}(\zeta_n, \zeta_{n+1})) - \tilde{\Psi}(\mathcal{P}(\zeta_n, \zeta_{n+1})) < \tilde{\Phi}(\mathcal{P}(\zeta_n, \zeta_{n+1})),$$

or equivalently

$$\mathcal{J}\mathcal{P}(\zeta_n, \zeta_{n+1}) \leq \mathcal{P}(\zeta_n, \zeta_{n+1}),$$

which is a contradiction. Hence, Equation (33) becomes

$$\mathcal{J}\mathcal{P}(\zeta_n, \zeta_{n+1}) \leq \mathcal{P}(\zeta_{n-1}, \zeta_n). \tag{34}$$

Therefore, $\lambda = \frac{1}{\mathcal{J}}$ from (29). By Lemma 3.1 of [19], and further from Equation (29), we obtain

$$\lim_{n \rightarrow +\infty} \mathcal{B}\theta_n = \lim_{n \rightarrow +\infty} \ell\theta_{n+1} = \mu, \mu \in \mathfrak{S}.$$

Furthermore, from condition (2), we have

$$\lim_{n \rightarrow +\infty} \mathcal{P}(\ell(\mathcal{B}\theta_n), \mathcal{B}(\ell\theta_n)) = 0,$$

and moreover, the continuity of \mathcal{B} and ℓ suggests that

$$\lim_{n \rightarrow +\infty} \ell(\mathcal{B}\theta_n) = \ell\mu, \text{ and } \lim_{n \rightarrow +\infty} \mathcal{B}(\ell\theta_n) = \mathcal{B}\mu.$$

Therefore,

$$\frac{1}{\mathcal{J}}\mathcal{P}(\mathcal{B}\mu, \ell\mu) \leq \mathcal{P}(\mathcal{B}\mu, \mathcal{B}(\ell\theta_n)) + \mathcal{J}\mathcal{P}(\mathcal{B}(\ell\theta_n), \ell(\mathcal{B}\theta_n)) + \mathcal{J}\mathcal{P}(\ell(\mathcal{B}\theta_n), \ell\mu). \tag{35}$$

So by letting $n \rightarrow +\infty$ in (35), we obtain that $\mathcal{P}(\mathcal{B}v, \ell v) = 0$, which implies that v is a coincidence point for the mappings \mathcal{B} and ℓ in \mathfrak{S} . \square

The following is a result obtained from Theorem 4 by relaxing the continuity property of ℓ and \mathcal{B} .

Theorem 5. *Suppose that the following conditions hold in Theorem 4:*

1. *A sequence $\{\ell\theta_n\} \subset \mathfrak{S}$ is a non-decreasing such that $\ell\theta_n \rightarrow \ell\theta \in \mathfrak{L}\mathfrak{S}$;*
2. *$\mathfrak{L}\mathfrak{S} \subseteq \mathfrak{S}$ is closed;*
3. *$\ell\theta_n \preceq \ell\theta$, for all $n \in \mathbb{N}$;*
4. *$\ell\theta \preceq \ell(\ell\theta)$;*
5. *$\ell\theta_0 \preceq \mathcal{B}\theta_0$ for some $\theta_0 \in \mathfrak{S}$.*

If \mathcal{B} and ℓ are the weakly compatible mappings, then \mathcal{B} and ℓ have a coincidence point. Furthermore, if \mathcal{B} and ℓ commute at their coincidence points, then \mathcal{B} and ℓ have a common fixed point in \mathfrak{S} .

Proof. From Theorem 4, there exists a Cauchy sequence $\{\zeta_n\} = \{\mathcal{B}\theta_n\} = \{\ell\theta_{n+1}\}$ in \mathfrak{S} . Thus, from the hypothesis, we have

$$\lim_{n \rightarrow +\infty} \mathcal{B}\theta_n = \lim_{n \rightarrow +\infty} \ell\theta_{n+1} = \ell\mu, \text{ for } \mu \in \mathfrak{S}.$$

Therefore, $\ell\theta_n \preceq \ell\mu, \forall n$. Now to claim that (\mathcal{B}, ℓ) have a coincidence point μ . From (24), we have

$$\tilde{\phi}(\delta \mathcal{P}(\mathcal{B}\theta_n, \mathcal{B}\theta)) \leq \tilde{\phi}(\mathcal{E}_\ell(\theta_n, \theta)) - \tilde{\psi}(\mathcal{E}_\ell(\theta_n, \theta)) + \Lambda\Omega(\mathcal{F}_\ell(\theta_n, \theta)), \tag{36}$$

where

$$\begin{aligned} \mathcal{E}_\ell(\theta_n, \mu) &= \max\left\{ \frac{\mathcal{P}(\ell\mu, \mathcal{B}\mu)[1 + \mathcal{P}(\ell\theta_n, \mathcal{B}\theta_n)]}{1 + \mathcal{P}(\ell\theta_n, \ell\mu)}, \frac{\mathcal{P}(\ell\theta_n, \mathcal{B}\theta_n) \mathcal{P}(\ell\mu, \mathcal{B}\mu)}{1 + \mathcal{P}(\ell\theta_n, \ell\mu)}, \right. \\ &\quad \left. \frac{\mathcal{P}(\ell\theta_n, \mathcal{B}\mu) + \mathcal{P}(\ell\mu, \mathcal{B}\theta_n)}{2\delta}, \mathcal{P}(\ell\theta_n, \ell\mu) \right\} \\ &\rightarrow \max\left\{ \mathcal{P}(\ell\mu, \mathcal{B}\mu), 0, \frac{d(\ell\mu, \mathcal{B}\mu)}{2\delta}, 0 \right\} \\ &= \mathcal{P}(\ell\mu, \mathcal{B}\mu) \text{ as } n \rightarrow +\infty, \end{aligned}$$

and

$$\begin{aligned} \mathcal{F}_\ell(\theta_n, \mu) &= \min\{\mathcal{P}(\ell\theta_n, \mathcal{B}\theta_n), \mathcal{P}(\ell\mu, \mathcal{B}\mu), \mathcal{P}(\ell\mu, \mathcal{B}\theta_n), \mathcal{P}(\ell\theta_n, \mathcal{B}\mu)\} \\ &\rightarrow \min\{0, \mathcal{P}(\ell\mu, \mathcal{B}\mu), 0, \mathcal{P}(\ell\mu, \mathcal{B}\mu)\} \\ &= 0 \text{ as } n \rightarrow +\infty. \end{aligned}$$

Letting $n \rightarrow +\infty$ in (36), we obtain

$$\tilde{\phi}(\delta \lim_{n \rightarrow +\infty} \mathcal{P}(\mathcal{B}\theta_n, \mathcal{B}\theta)) \leq \tilde{\phi}(\mathcal{P}(\ell\mu, \mathcal{B}\mu)) - \tilde{\psi}(\mathcal{P}(\ell\mu, \mathcal{B}\mu)) < \tilde{\phi}(\mathcal{P}(\ell\mu, \mathcal{B}\mu)). \tag{37}$$

Furthermore, from the property of $\tilde{\phi}$, we obtain

$$\lim_{n \rightarrow +\infty} \mathcal{P}(\mathcal{B}\theta_n, \mathcal{B}\theta) < \frac{1}{\delta} \mathcal{P}(\ell\mu, \mathcal{B}\mu). \tag{38}$$

Furthermore, the triangular inequality of \mathcal{P} implies that

$$\frac{1}{\delta} \mathcal{P}(\ell\mu, \mathcal{B}\mu) \leq \mathcal{P}(\ell\mu, \mathcal{B}\theta_n) + \mathcal{P}(\mathcal{B}\theta_n, \mathcal{B}\mu). \tag{39}$$

If $\ell\mu \neq \mathcal{B}\mu$, then (38) and (39) lead to a contradiction. Therefore, $\ell\mu = \mathcal{B}\mu$. Assume that $\ell\mu = \mathcal{B}\mu = \rho$, then $\mathcal{B}\rho = \mathcal{B}(\ell\mu) = \ell(\mathcal{B}\mu) = \ell\rho$. Since $\ell\mu \preceq \ell(\ell\mu) = \ell\rho$, then by Equation (36) with $\ell\mu = \mathcal{B}\mu$ and $\ell\rho = \mathcal{B}\rho$, we have

$$\tilde{\phi}(s\mathcal{P}(\mathcal{B}\mu, \mathcal{B}\rho)) \leq \tilde{\phi}(\mathcal{E}_\ell(\mu, \rho)) - \tilde{\psi}(\mathcal{E}_\ell(\mu, \rho)) < \tilde{\phi}(\mathcal{P}(\mathcal{B}\mu, \mathcal{B}\rho)), \tag{40}$$

or equivalently,

$$s\mathcal{P}(\mathcal{B}\mu, \mathcal{B}\rho) \leq \mathcal{P}(\mathcal{B}\mu, \mathcal{B}\rho),$$

which shows a contradiction, if $\mathcal{B}\mu \neq \mathcal{B}\rho$. Therefore, $\mathcal{B}\mu = \mathcal{B}\rho = \rho$ which suggests that $\mathcal{B}\mu = \ell\rho = \rho$. This completes the result. \square

Definition 7. A mapping $\mathcal{B} : \mathfrak{S} \times \mathfrak{S} \rightarrow \mathfrak{S}$ is a generalized $(\tilde{\phi}, \tilde{\psi})$ -contractive mapping over a b-metric space \mathfrak{S} with respect to a self mapping ℓ on \mathfrak{S} , if it satisfies the following condition:

$$\phi(s^k \mathcal{P}(\mathcal{B}(\theta, \zeta), \mathcal{B}(\varrho, \sigma))) \leq \tilde{\phi}(\mathcal{E}_\ell(\theta, \zeta, \varrho, \sigma)) - \tilde{\psi}(\mathcal{E}_\ell(\theta, \zeta, \varrho, \sigma)) + \Lambda\Omega(\mathcal{F}_\ell(\theta, \zeta, \varrho, \sigma)), \tag{41}$$

for all $\theta, \zeta, \varrho, \sigma \in \mathfrak{S}$ such that $\ell\theta \preceq \ell\varrho$ and $\ell\zeta \succeq \ell\sigma$, $k > 2$, $s > 1$, $\tilde{\phi} \in \tilde{\Phi}$, $\tilde{\psi} \in \tilde{\Psi}$, $\Omega \in \Theta$ and where

$$\mathcal{E}_\ell(\theta, \zeta, \varrho, \sigma) = \max\left\{ \frac{\mathcal{P}(\ell\varrho, \mathcal{B}(\varrho, \sigma))[1 + \mathcal{P}(\ell\theta, \mathcal{B}(\theta, \zeta))]}{1 + \mathcal{P}(\ell\theta, \ell\varrho)}, \frac{\mathcal{P}(\ell\theta, \mathcal{B}(\theta, \zeta)) \mathcal{P}(\ell\varrho, \mathcal{B}(\varrho, \sigma))}{1 + \mathcal{P}(\ell\theta, \ell\varrho)}, \right. \\ \left. \frac{\mathcal{P}(\ell\theta, \mathcal{B}(\varrho, \sigma)) + \mathcal{P}(\ell\varrho, \mathcal{B}(\theta, \zeta))}{2s}, \mathcal{P}(\ell\theta, \ell\varrho) \right\} \tag{42}$$

and

$$\mathcal{F}_\ell(\theta, \zeta, \varrho, \sigma) = \min\{\mathcal{P}(\ell\theta, \mathcal{B}(\theta, \zeta)), \mathcal{P}(\ell\varrho, \mathcal{B}(\varrho, \sigma)), \mathcal{P}(\ell\varrho, \mathcal{B}(\theta, \zeta)), \mathcal{P}(\ell\theta, \mathcal{B}(\varrho, \sigma))\}. \tag{43}$$

Theorem 6. Let the mapping $\mathcal{B} : \mathfrak{S} \times \mathfrak{S} \rightarrow \mathfrak{S}$ be a generalized $(\tilde{\phi}, \tilde{\psi})$ -contractive mapping with respect to a self mapping ℓ on c.p.o.b-m.s. \mathfrak{S} . Assume that the mappings \mathcal{B} and ℓ are continuous, \mathcal{B} has mixed ℓ -monotone property and commutes with ℓ . If for some $(\theta_0, \zeta_0) \in \mathfrak{S} \times \mathfrak{S}$ with $\ell\theta_0 \preceq \mathcal{B}(\theta_0, \zeta_0)$, $\ell\zeta_0 \succeq \mathcal{B}(\zeta_0, \theta_0)$ and $\mathcal{B}(\mathfrak{S} \times \mathfrak{S}) \subseteq \ell(\mathfrak{S})$, then the mappings \mathcal{B} and ℓ have a coupled coincidence point in \mathfrak{S} .

Proof. There exist two sequences $\{\theta_n\}$ and $\{\zeta_n\}$ in \mathfrak{S} from Theorem 2.2 of [14] such that

$$\ell\theta_{n+1} = \mathcal{B}(\theta_n, \zeta_n), \quad \ell\zeta_{n+1} = \mathcal{B}(\zeta_n, \theta_n), \text{ for all } n \geq 0,$$

where the sequence $\{\ell\theta_n\}$ is non-decreasing and $\{\ell\zeta_n\}$ is non-increasing in \mathfrak{S} . Suppose $\theta = \theta_n, \zeta = \zeta_n, \varrho = \theta_{n+1}, \sigma = \zeta_{n+1}$ in (41), then Equation (41) becomes

$$\tilde{\phi}(s^k \mathcal{P}(\ell\theta_{n+1}, \ell\theta_{n+2})) = \tilde{\phi}(s^k \mathcal{P}(\mathcal{B}(\theta_n, \zeta_n), \mathcal{B}(\theta_{n+1}, \zeta_{n+1}))) \\ \leq \tilde{\phi}(\mathcal{E}_\ell(\theta_n, \zeta_n, \theta_{n+1}, \zeta_{n+1})) - \tilde{\psi}(\mathcal{E}_\ell(\theta_n, \zeta_n, \theta_{n+1}, \zeta_{n+1})) \\ + \Lambda\Omega(\mathcal{F}_\ell(\theta_n, \zeta_n, \theta_{n+1}, \zeta_{n+1})), \tag{44}$$

where

$$\mathcal{E}_\ell(\theta_n, \zeta_n, \theta_{n+1}, \zeta_{n+1}) \leq \max\{\mathcal{P}(\ell\theta_n, \ell\theta_{n+1}), \mathcal{P}(\ell\theta_{n+1}, \ell\theta_{n+2})\}$$

and

$$\mathcal{F}_\ell(\theta_n, \zeta_n, \theta_{n+1}, \zeta_{n+1}) = \min\{\mathcal{P}(\ell\theta_n, \mathcal{B}(\theta_n, \zeta_n)), \mathcal{P}(\ell\theta_{n+1}, \mathcal{B}(\theta_{n+1}, \zeta_{n+1})), \\ \mathcal{P}(\ell\theta_n, \mathcal{B}(\theta_{n+1}, \zeta_{n+1})), \mathcal{P}(\ell\theta_{n+1}, \mathcal{B}(\theta_n, \zeta_n))\} \\ = 0.$$

Therefore from the Equation (44), we obtain

$$\begin{aligned} \tilde{\phi}(j^k \rho(\ell\theta_{n+1}, \ell\theta_{n+2})) &\leq \tilde{\phi}(\max\{\rho(\ell\theta_n, \ell\theta_{n+1}), \rho(\ell\theta_{n+1}, \ell\theta_{n+2})\}) \\ &\quad \tilde{\psi}(\max\{\rho(\ell\theta_n, \ell\theta_{n+1}), \rho(\ell\theta_{n+1}, \ell\theta_{n+2})\}). \end{aligned} \tag{45}$$

Similarly, by taking $\theta = \zeta_n, \zeta = \theta_n, \varrho = \zeta_{n+1}, \sigma = \theta_{n+1}$ in (41), we arrive at

$$\begin{aligned} \tilde{\phi}(j^k \rho(\ell\zeta_{n+1}, \ell\zeta_{n+2})) &\leq \tilde{\phi}(\max\{\rho(\ell\zeta_n, \ell\zeta_{n+1}), \rho(\ell\zeta_{n+1}, \ell\zeta_{n+2})\}) \\ &\quad \tilde{\psi}(\max\{\rho(\ell\zeta_n, \ell\zeta_{n+1}), \rho(\ell\zeta_{n+1}, \ell\zeta_{n+2})\}). \end{aligned} \tag{46}$$

As by the result of $\max\{\tilde{\phi}(a_1), \tilde{\phi}(a_2)\} = \tilde{\phi}\{\max\{a_1, a_2\}\}$ for $a_1, a_2 \in [0, +\infty)$, the Equations (45) and (46) in turn imply that

$$\begin{aligned} \tilde{\phi}(j^k \kappa_n) &\leq \tilde{\phi}(\max\{\rho(\ell\theta_n, \ell\theta_{n+1}), \rho(\ell\theta_{n+1}, \ell\theta_{n+2}), \rho(\ell\zeta_n, \ell\zeta_{n+1}), \rho(\ell\zeta_{n+1}, \ell\zeta_{n+2})\}) \\ &\quad - \tilde{\psi}(\max\{\rho(\ell\theta_n, \ell\theta_{n+1}), \rho(\ell\theta_{n+1}, \ell\theta_{n+2}), \rho(\ell\zeta_n, \ell\zeta_{n+1}), \rho(\ell\zeta_{n+1}, \ell\zeta_{n+2})\}) \end{aligned} \tag{47}$$

where

$$\kappa_n = \max\{\rho(\ell\theta_{n+1}, \ell\theta_{n+2}), \rho(\ell\zeta_{n+1}, \ell\zeta_{n+2})\}. \tag{48}$$

Notate

$$\Sigma_n = \max\{\rho(\ell\theta_n, \ell\theta_{n+1}), \rho(\ell\theta_{n+1}, \ell\theta_{n+2}), \rho(\ell\zeta_n, \ell\zeta_{n+1}), \rho(\ell\zeta_{n+1}, \ell\zeta_{n+2})\}, \tag{49}$$

then from Equations (45)–(48), we obtain

$$j^k \kappa_n \leq \Sigma_n. \tag{50}$$

Next to show that

$$\kappa_n \leq \lambda \kappa_{n-1}, (n \geq 1) \tag{51}$$

where $\lambda = \frac{1}{j^k}$.

It is evident that $j^k \kappa_n \leq \kappa_n$, if $\Sigma_n = \kappa_n$ from (50). Therefore, $\kappa_n = 0$ as $j > 1$ and hence (51) holds. Furthermore, if $\Sigma_n = \max\{\rho(\ell\theta_n, \ell\theta_{n+1}), \rho(\ell\zeta_n, \ell\zeta_{n+1})\}$, i.e., $\Sigma_n = \kappa_{n-1}$ then (50) follows (51). Therefore, we obtain $\kappa_n \leq \lambda^n \kappa_0$ from (50). Hence, we obtain

$$\rho(\ell\theta_{n+1}, \ell\theta_{n+2}) \leq \lambda^n \kappa_0 \text{ and } \rho(\ell\zeta_{n+1}, \ell\zeta_{n+2}) \leq \lambda^n \kappa_0, \tag{52}$$

and then from Lemma 3.1. of [19], it is clear that $\{\ell\theta_n\}$ and $\{\ell\zeta_n\}$ in \mathfrak{S} are Cauchy sequences. Therefore, by continuous of the mappings \mathcal{B} and ℓ , we conclude that mappings \mathcal{B} and ℓ have a coupled coincidence point in \mathfrak{S} . \square

Corollary 3. Suppose that a continuous mapping $\mathcal{B} : \mathfrak{S} \times \mathfrak{S} \rightarrow \mathfrak{S}$ has the property of mixed monotone over the c.p.o.b-m.s. $(\mathfrak{S}, \rho, \preceq)$. If $\theta_0 \preceq \mathcal{B}(\theta_0, \zeta_0)$ and $\zeta_0 \succeq \mathcal{B}(\zeta_0, \theta_0)$, for certain $(\theta_0, \zeta_0) \in \mathfrak{S} \times \mathfrak{S}$, then \mathcal{B} has a coupled fixed point in \mathfrak{S} with the following contraction conditions:

(i)

$$\tilde{\phi}(j^k \rho(\mathcal{B}(\theta, \zeta), \mathcal{B}(\varrho, \sigma))) \leq \tilde{\phi}(\mathcal{E}_\ell(\theta, \zeta, \varrho, \sigma)) - \tilde{\psi}(\mathcal{E}_\ell(\theta, \zeta, \varrho, \sigma)), \tag{53}$$

(ii)

$$\rho(\mathcal{B}(\theta, \zeta), \mathcal{B}(\varrho, \sigma)) \leq \frac{1}{j^k} \mathcal{E}_\ell(\theta, \zeta, \varrho, \sigma) - \frac{1}{j^k} \tilde{\psi}(\mathcal{E}_\ell(\theta, \zeta, \varrho, \sigma)), \tag{54}$$

where

$$\begin{aligned} \mathcal{E}_\ell(\theta, \zeta, \varrho, \sigma) &= \max\left\{ \frac{\rho(\varrho, \mathcal{B}(\varrho, \sigma))[1 + \rho(\theta, \mathcal{B}(\theta, \zeta))]}{1 + \rho(\theta, \varrho)}, \frac{\rho(\theta, \mathcal{B}(\theta, \zeta)) \rho(\varrho, \mathcal{B}(\varrho, \sigma))}{1 + \rho(\theta, \varrho)}, \right. \\ &\quad \left. \frac{\rho(\theta, \mathcal{B}(\varrho, \sigma)) + \rho(\varrho, \mathcal{B}(\theta, \zeta))}{2j}, \rho(\theta, \varrho) \right\} \end{aligned}$$

and

$$\mathcal{F}_\ell(\theta, \zeta) = \min\{\rho(\theta, \mathcal{B}(\theta, \zeta)), \rho(\varrho, \mathcal{B}(\varrho, \sigma)), \rho(\varrho, \mathcal{B}(\theta, \zeta)), \rho(\theta, \mathcal{B}(\varrho, \sigma))\},$$

for all $\theta, \zeta, \varrho, \sigma \in \mathfrak{S}$ with $\theta \preceq \varrho$ and $\zeta \succeq \sigma$, $k > 2$, $s > 1$, $\tilde{\phi} \in \tilde{\Phi}$ and $\tilde{\psi} \in \tilde{\Psi}$.

Theorem 7. If in Theorem 6, $(\mathcal{B}(j^*, h^*), \mathcal{B}(h^*, j^*))$ is comparable to $(\mathcal{B}(\theta, \zeta), \mathcal{B}(\zeta, \theta))$ and $(\mathcal{B}(k, c), \mathcal{B}(c, k))$, for all $(\theta, \zeta), (k, c) \in \mathfrak{S} \times \mathfrak{S}$ and some $(j^*, h^*) \in \mathfrak{S} \times \mathfrak{S}$, then the mappings \mathcal{B} and ℓ have a unique coupled common fixed point in \mathfrak{S} .

Proof. From Theorem 6, the mappings \mathcal{B} and ℓ have a coupled coincidence point in \mathfrak{S} . Assume that two coupled coincidence points (θ, ζ) and (k, c) for \mathcal{B}, ℓ exist in \mathfrak{S} . Then we have to show that $\ell\theta = \ell k$ and $\ell\zeta = \ell c$. From the hypotheses for $(j^*, h^*) \in \mathfrak{S} \times \mathfrak{S}$, $(\mathcal{B}(j^*, h^*), \mathcal{B}(h^*, j^*))$ is comparable to $(\mathcal{B}(\theta, \zeta), \mathcal{B}(\zeta, \theta))$ and $(\mathcal{B}(k, c), \mathcal{B}(c, k))$.

Suppose

$$\begin{aligned} (\mathcal{B}(\theta, \zeta), \mathcal{B}(\zeta, \theta)) &\preceq (\mathcal{B}(j^*, h^*), \mathcal{B}(h^*, j^*)) \text{ and} \\ (\mathcal{B}(k, c), \mathcal{B}(c, k)) &\preceq (\mathcal{B}(j^*, h^*), \mathcal{B}(h^*, j^*)). \end{aligned}$$

Let $j^*_0 = j^*$ and $h^*_0 = h^*$; then, there is a point $(j^*_1, h^*_1) \in \mathfrak{S} \times \mathfrak{S}$ such that

$$\ell j^*_1 = \mathcal{B}(j^*_0, h^*_0), \ell h^*_1 = \mathcal{B}(h^*_0, j^*_0), (n \geq 1).$$

By induction, there exist two sequences $\{\ell j^*_n\}, \{\ell h^*_n\}$ in \mathfrak{S} with

$$\ell j^*_{n+1} = \mathcal{B}(j^*_n, h^*_n), \ell h^*_{n+1} = \mathcal{B}(h^*_n, j^*_n), (n \geq 0).$$

Furthermore, by letting $\theta_0 = \theta, \zeta_0 = \zeta$ and $k_0 = k, c_0 = c$, there will be other sequences $\{\ell\theta_n\}, \{\ell\zeta_n\}$ and $\{\ell k_n\}, \{\ell c_n\}$ in \mathfrak{S} such that

$$\ell\theta_n \rightarrow \mathcal{B}(\theta, \zeta), \ell\zeta_n \rightarrow \mathcal{B}(\zeta, \theta), \ell k_n \rightarrow \mathcal{B}(k, c), \ell c_n \rightarrow \mathcal{B}(c, k) (n \geq 1). \tag{55}$$

Thus, by induction, we obtain

$$(\ell\theta_n, \ell\zeta_n) \leq (\ell j^*_n, \ell h^*_n), (n \geq 0). \tag{56}$$

From (41)

$$\begin{aligned} \tilde{\phi}(\rho(\ell\theta, \ell j^*_{n+1})) &\leq \tilde{\phi}(s^k \rho(\ell\theta, \ell j^*_{n+1})) = \tilde{\phi}(s^k \rho(\mathcal{B}(\theta, \zeta), \mathcal{B}(j^*_n, h^*_n))) \\ &\leq \tilde{\phi}(\mathcal{E}_\ell(\theta, \zeta, j^*_n, h^*_n)) - \tilde{\psi}(\mathcal{E}_\ell(\theta, \zeta, j^*_n, h^*_n)) + \Lambda\Omega(\mathcal{F}_\ell(\theta, \zeta, j^*_n, h^*_n)), \end{aligned} \tag{57}$$

where

$$\begin{aligned} \mathcal{F}_\ell(\theta, \zeta, j^*_n, h^*_n) &= \max\left\{ \frac{\rho(j^*_n, \mathcal{B}(j^*_n, h^*_n)) [1 + \rho(\ell\theta, \mathcal{B}(\theta, \zeta))]}{1 + \rho(\ell\theta, \ell j^*_n)}, \right. \\ &\quad \frac{\rho(\ell\theta, \mathcal{B}(\theta, \zeta)) \rho(j^*_n, \mathcal{B}(j^*_n, h^*_n))}{1 + \rho(\ell\theta, j^*_n)}, \\ &\quad \left. \frac{\rho(\ell\theta, \mathcal{B}(j^*_n, h^*_n)) + \rho(j^*_n, \mathcal{B}(\theta, \zeta))}{2s}, \rho(\ell\theta, \ell j^*_n) \right\} \\ &= \max\left\{ 0, \frac{\rho(\ell\theta, j^*_n)}{s}, \rho(\ell\theta, \ell j^*_n) \right\} \\ &= \rho(\ell\theta, \ell j^*_n) \end{aligned}$$

and

$$\begin{aligned} \mathcal{F}_\ell(\theta, \zeta, j^*_n, h^*_n) &= \min\{\rho(\ell\theta, \mathcal{B}(\theta, \zeta)), \rho(\ell j^*_n, \mathcal{B}(\ell j^*_n, h^*_n)), \rho(\ell j^*_n, \mathcal{B}(\theta, \zeta)), \\ &\quad \rho(\ell\theta, \mathcal{B}(j^*_n, h^*_n))\} \\ &= 0. \end{aligned}$$

Hence, from Equation (57), we obtain

$$\tilde{\phi}(\rho(\ell\theta, \ell\mathcal{I}^*_{n+1})) \leq \tilde{\phi}(\rho(\ell\theta, \ell\mathcal{I}^*_n)) - \tilde{\psi}(\rho(\ell\theta, \ell\mathcal{I}^*_n)). \tag{58}$$

Furthermore, using a similar manner we obtain

$$\tilde{\phi}(\rho(\ell\zeta, \ell\mathcal{H}^*_{n+1})) \leq \tilde{\phi}(\rho(\ell\zeta, \ell\mathcal{H}^*_n)) - \tilde{\psi}(\rho(\ell\zeta, \ell\mathcal{H}^*_n)). \tag{59}$$

From Equations (58) and (59), we have

$$\begin{aligned} \tilde{\phi}(\max\{\rho(\ell\theta, \ell\mathcal{I}^*_{n+1}), \rho(\ell\zeta, \ell\mathcal{H}^*_{n+1})\}) &\leq \tilde{\phi}(\max\{\rho(\ell\theta, \ell\mathcal{I}^*_n), \rho(\ell\zeta, \ell\mathcal{H}^*_n)\}) \\ &\quad - \tilde{\psi}(\max\{\rho(\ell\theta, \ell\mathcal{I}^*_n), \rho(\ell\zeta, \ell\mathcal{H}^*_n)\}) \tag{60} \\ &< \tilde{\phi}(\max\{\rho(\ell\theta, \ell\mathcal{I}^*_n), \rho(\ell\zeta, \ell\mathcal{H}^*_n)\}). \end{aligned}$$

Furthermore, the property of $\tilde{\phi}$, Equation (60) implies that

$$\max\{\rho(\ell\theta, \ell\mathcal{I}^*_{n+1}), \rho(\ell\zeta, \ell\mathcal{H}^*_{n+1})\} < \max\{\rho(\ell\theta, \ell\mathcal{I}^*_n), \rho(\ell\zeta, \ell\mathcal{H}^*_n)\}.$$

Therefore, $\max\{\rho(\ell\theta, \ell\mathcal{I}^*_n), \rho(\ell\zeta, \ell\mathcal{H}^*_n)\}$ is a decreasing sequence of positive reals and bounded below. Therefore, we have

$$\lim_{n \rightarrow +\infty} \max\{\rho(\ell\theta, \ell\mathcal{I}^*_n), \rho(\ell\zeta, \ell\mathcal{H}^*_n)\} = \mathcal{X}, \mathcal{X} \geq 0. \tag{61}$$

Letting $n \rightarrow +\infty$ in Equation (60), we obtain

$$\tilde{\phi}(\mathcal{X}) \leq \tilde{\phi}(\mathcal{X}) - \tilde{\psi}(\mathcal{X}), \tag{62}$$

and also by the property of $\tilde{\phi}$, we obtained that $\tilde{\psi}(\mathcal{X}) = 0$ and hence, $\mathcal{X} = 0$. Therefore Equation (61) follows that

$$\lim_{n \rightarrow +\infty} \max\{\rho(\ell\theta, \ell\mathcal{I}^*_n), \rho(\ell\zeta, \ell\mathcal{H}^*_n)\} = 0,$$

which implies that

$$\lim_{n \rightarrow +\infty} \rho(\ell\theta, \ell\mathcal{I}^*_n) = 0 \text{ and } \lim_{n \rightarrow +\infty} \rho(\ell\zeta, \ell\mathcal{H}^*_n) = 0. \tag{63}$$

Again by similar process, we obtain that

$$\lim_{n \rightarrow +\infty} \rho(\ell\mathcal{I}, \ell\mathcal{I}^*_n) = 0 \text{ and } \lim_{n \rightarrow +\infty} \rho(\ell\mathcal{C}, \ell\mathcal{H}^*_n) = 0. \tag{64}$$

Therefore from Equations (63) and (64), we have $\ell\theta = \ell\mathcal{I}$ and $\ell\zeta = \ell\mathcal{C}$. Since $\ell\theta = \mathcal{B}(\theta, \zeta)$ and $\ell\zeta = \mathcal{B}(\zeta, \theta)$, and there is the commutativity property of \mathcal{B} and ℓ , we have

$$\ell(\ell\theta) = \ell(\mathcal{B}(\theta, \zeta)) = \mathcal{B}(\ell\theta, \ell\zeta) \text{ and } \ell(\ell\zeta) = \ell(\mathcal{B}(\zeta, \theta)) = \mathcal{B}(\ell\zeta, \ell\theta). \tag{65}$$

Suppose $\ell\theta = \mathcal{I}^*_\vartheta$ and $\ell\zeta = \mathcal{H}^*_\vartheta$, then from Equation (65), we obtain

$$\ell(\mathcal{I}^*_\vartheta) = \mathcal{B}(\mathcal{I}^*_\vartheta, \mathcal{H}^*_\vartheta) \text{ and } \ell(\mathcal{H}^*_\vartheta) = \mathcal{B}(\mathcal{H}^*_\vartheta, \mathcal{I}^*_\vartheta), \tag{66}$$

which shows that \mathcal{B}, ℓ have a coupled coincidence point $(\mathcal{I}^*_\vartheta, \mathcal{H}^*_\vartheta)$. Thus, $\ell(\mathcal{I}^*_\vartheta) = \ell\mathcal{I}$ and $\ell(\mathcal{H}^*_\vartheta) = \ell\mathcal{C}$; hence, $\ell(\mathcal{I}^*_\vartheta) = \mathcal{I}^*_\vartheta$ and $\ell(\mathcal{H}^*_\vartheta) = \mathcal{H}^*_\vartheta$. Therefore, from (66), $(\mathcal{I}^*_\vartheta, \mathcal{H}^*_\vartheta)$ is a coupled common fixed point of \mathcal{B} and ℓ .

If $(\mathcal{I}^*_\vartheta, \mathcal{H}^*_\vartheta)$ is another coupled common fixed point of \mathcal{B} and ℓ . Then, $\mathcal{I}^*_\vartheta = \ell\mathcal{I}^*_\vartheta = \mathcal{B}(\mathcal{I}^*_\vartheta, \mathcal{H}^*_\vartheta)$ and $\mathcal{H}^*_\vartheta = \ell\mathcal{H}^*_\vartheta = \mathcal{B}(\mathcal{H}^*_\vartheta, \mathcal{I}^*_\vartheta)$. As $(\mathcal{I}^*_\vartheta, \mathcal{H}^*_\vartheta)$ is a coupled common fixed point of

\mathcal{B} and ℓ then $\ell \mathcal{I}_{\mathcal{A}}^* = \ell \theta = \mathcal{I}_{\mathcal{B}}^*$ and $\ell \mathcal{H}_{\mathcal{A}}^* = \ell \zeta = \mathcal{H}_{\mathcal{B}}^*$. Therefore, $\mathcal{I}_{\mathcal{A}}^* = \ell \mathcal{I}_{\mathcal{A}}^* = \ell \mathcal{I}_{\mathcal{B}}^* = \mathcal{I}_{\mathcal{B}}^*$ and $\mathcal{H}_{\mathcal{A}}^* = \ell \mathcal{H}_{\mathcal{A}}^* = \ell \mathcal{H}_{\mathcal{B}}^* = \mathcal{H}_{\mathcal{B}}^*$. This completes the proof. \square

Theorem 8. *If $\ell \theta_0 \preceq \ell \zeta_0$ or $\ell \theta_0 \succeq \ell \zeta_0$, then a unique common fixed point for the mappings \mathcal{B} and ℓ exists in \mathfrak{S} of Theorem 7.*

Proof. We have to claim that $\theta = \zeta$ for a unique coupled common fixed point (θ, ζ) of the mappings \mathcal{B} and ℓ in \mathfrak{S} . By induction, we obtain that $\ell \theta_n \preceq \ell \zeta_n, (n \geq 0)$ when $\ell \theta_0 \preceq \ell \zeta_0$. Therefore, by following Lemma 2 of [20], we obtain that

$$\begin{aligned} \tilde{\phi}(j^{k-2} \rho(\theta, \zeta)) &= \tilde{\phi}(j^k \frac{1}{j^2} \rho(\theta, \zeta)) \leq \limsup_{n \rightarrow +\infty} \tilde{\phi}(j^k \rho(\theta_{n+1}, \zeta_{n+1})) \\ &= \limsup_{n \rightarrow +\infty} \tilde{\phi}(j^k \rho(\mathcal{B}(\theta_n, \zeta_n), \mathcal{B}(\zeta_n, \theta_n))) \\ &\leq \limsup_{n \rightarrow +\infty} \tilde{\phi}(\mathcal{E}_{\ell}(\theta_n, \zeta_n, \zeta_n, \theta_n)) - \liminf_{n \rightarrow +\infty} \tilde{\psi}(\mathcal{E}_{\ell}(\theta_n, \zeta_n, \zeta_n, \theta_n)) \\ &\quad + \Lambda \limsup_{n \rightarrow +\infty} \Omega(\mathcal{F}_{\ell}(\theta_n, \zeta_n, \zeta_n, \theta_n)) \\ &\leq \tilde{\phi}(\rho(\theta, \zeta)) - \liminf_{n \rightarrow +\infty} \tilde{\psi}(\mathcal{E}_f(\theta_n, \zeta_n, \zeta_n, \theta_n)) \\ &< \tilde{\phi}(\rho(\theta, \zeta)), \end{aligned} \tag{67}$$

which is a contradiction form of Equation (67). Therefore, $\theta = \zeta$.

A similar proof can also see the same conclusion if $\ell \theta_0 \succeq \ell \zeta_0$. \square

Remark 1. *By following [4], the condition*

$$\tilde{\phi}(\rho(\mathcal{B}(\theta, \zeta), \mathcal{B}(\varrho, \varsigma))) \leq \tilde{\phi}(\max\{\rho(\ell \theta, \ell \varrho), \rho(\ell \zeta, \ell \varsigma)\}) - \tilde{\psi}(\max\{\rho(\ell \theta, \ell \varrho), \rho(\ell \zeta, \ell \varsigma)\})$$

is equivalent to

$$\rho(\mathcal{B}(\theta, \zeta), \mathcal{B}(\varrho, \varsigma)) \leq \varphi(\max\{\rho(\ell \theta, \ell \varrho), \rho(\ell \zeta, \ell \varsigma)\}),$$

when $s = 1$ and where φ is a continuous self mapping on $[0, +\infty)$ with $\varphi(a) < a$, for all $a > 0$ and $\varphi(a) = 0$ if and only if $a = 0$ and, $\tilde{\phi} \in \tilde{\Phi}, \tilde{\psi} \in \tilde{\Psi}$. Hence, the results obtained in this paper are generalizing and extending the results of [22] and many comparable results in the literature.

We illustrate some examples based on the metric as follows.

Example 1. *Let $\mathfrak{S} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, and define a metric $\rho : \mathfrak{S} \times \mathfrak{S} \rightarrow \mathfrak{S}$ by*

$$\begin{aligned} \rho(\theta, \zeta) &= \rho(\zeta, \theta) = 0, \text{ if } \theta = \zeta = x_1, x_2, x_3, x_4, x_5, x_6 \text{ and } \theta = \zeta; \\ \rho(\theta, \zeta) &= \rho(\zeta, \theta) = 3, \text{ if } \theta = \zeta = x_1, x_2, x_3, x_4, x_5 \text{ and } \theta \neq \zeta; \\ \rho(\theta, \zeta) &= \rho(\zeta, \theta) = 12, \text{ if } \theta = x_1, x_2, x_3, x_4 \text{ and } \zeta = x_6; \\ \rho(\theta, \zeta) &= \rho(\zeta, \theta) = 20, \text{ if } \theta = x_5 \text{ and } \zeta = x_6, \text{ with usual order } \leq. \end{aligned}$$

If \mathcal{B} is a self mapping on \mathfrak{S} with $\mathcal{B}x_1 = \mathcal{B}x_2 = \mathcal{B}x_3 = \mathcal{B}x_4 = \mathcal{B}x_5 = 1, \mathcal{B}x_6 = 2$, then \mathcal{B} has a fixed point in \mathfrak{S} with the distance functions $\tilde{\phi}(a) = \frac{a}{2}$ and $\tilde{\psi}(a) = \frac{a}{4}$, for all $a \in [0, +\infty)$.

Proof. For $s = 2$, $(\mathfrak{S}, \rho, \leq)$ is a c.p.o.b-m.s. If $\theta < \zeta$ for some $\theta, \zeta \in \mathfrak{S}$, then we have the cases below.

Case (a). If $\theta, \zeta \in \{x_1, x_2, x_3, x_4, x_5\}$ then $\rho(\mathcal{B}\theta, \mathcal{B}\zeta) = \rho(x_1, x_1) = 0$. Hence,

$$\tilde{\phi}(2\rho(\mathcal{B}\theta, \mathcal{B}\zeta)) = 0 \leq \tilde{\phi}(\mathcal{E}(\theta, \zeta)) - \tilde{\psi}(\mathcal{E}(\theta, \zeta)).$$

Case (b). If $\theta \in \{x_1, x_2, x_3, x_4, x_5\}$ and $\zeta = x_6$, then $\rho(\mathcal{B}\theta, \mathcal{B}\zeta) = \rho(x_1, x_2) = 3$, $\mathcal{E}(x_6, x_5) = 20$ and $\mathcal{E}(\theta, x_6) = 12$, for $\theta \in \{x_1, x_2, x_3, x_4\}$. Therefore,

$$\tilde{\phi}(2\rho(\mathcal{B}\theta, \mathcal{B}\zeta)) \leq \frac{\mathcal{E}(\theta, \zeta)}{d} = \tilde{\phi}(\mathcal{E}(\theta, \zeta)) - \tilde{\psi}(\mathcal{E}(\theta, \zeta)).$$

Hence all assumptions of Corollary 1 are satisfied; hence \mathcal{B} has a fixed point in \mathfrak{S} . \square

Example 2. Let us define a metric ρ on $\mathfrak{S} = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots\}$ with the usual order \leq by

$$\rho(\theta, \zeta) = \begin{cases} 0, & \text{if } \theta = \zeta, \\ 1, & \text{if } \theta \neq \zeta \in \{0, 1\}, \\ |\theta - \zeta|, & \text{if } \theta, \zeta \in \{0, \frac{1}{2n}, \frac{1}{2m} : n \neq m, n \geq 1, m \geq 1\}, \\ 2, & \text{otherwise.} \end{cases}$$

If \mathcal{B} on \mathfrak{S} is a self mapping such that $\mathcal{B}0 = 0, \mathcal{B}\frac{1}{n} = \frac{1}{12n}$, ($n \geq 1$), then \mathcal{B} has a fixed point in \mathfrak{S} with the distance functions $\tilde{\phi}(a) = a$ and $\tilde{\psi}(a) = \frac{4a}{5}$, for all $a \in [0, +\infty)$.

Proof. By definition, a metric ρ is discontinuous. Furthermore, for $s = \frac{12}{5}$, $(\mathfrak{S}, \rho, \leq)$ is a c.p.o.b-m.s. Now we will have the following cases for $\theta, \zeta \in \mathfrak{S}$ with $\theta < \zeta$:

Case (a). If $\theta = 0$ and $\zeta = \frac{1}{n}$ ($n > 0$), then $\rho(\mathcal{B}\theta, \mathcal{B}\zeta) = \rho(0, \frac{1}{12n}) = \frac{1}{12n}$ and $\mathcal{E}(\theta, \zeta) = \frac{1}{n}$, $\mathcal{E}(\theta, \zeta) = \{1, 2\}$. Therefore,

$$\tilde{\phi}\left(\frac{12}{5}\rho(\mathcal{B}\theta, \mathcal{B}\zeta)\right) \leq \frac{\mathcal{E}(\theta, \zeta)}{5} = \tilde{\phi}(\mathcal{E}(\theta, \zeta)) - \tilde{\psi}(\mathcal{E}(\theta, \zeta)).$$

Case (b). If $\theta = \frac{1}{m}$ and $\zeta = \frac{1}{n}$ for $m > n \geq 1$, then

$$\rho(\mathcal{B}\theta, \mathcal{B}\zeta) = \rho\left(\frac{1}{12m}, \frac{1}{12n}\right) \text{ and } \mathcal{E}(\theta, \zeta) = \geq \frac{1}{n} - \frac{1}{m} \text{ or } \mathcal{E}(\theta, \zeta) = 2.$$

Therefore,

$$\tilde{\phi}\left(\frac{12}{5}\rho(\mathcal{B}\theta, \mathcal{B}\zeta)\right) \leq \frac{\mathcal{E}(\theta, \zeta)}{5} = \tilde{\phi}(\mathcal{E}(\theta, \zeta)) - \tilde{\psi}(\mathcal{E}(\theta, \zeta)).$$

As all assumptions of Corollary 1 are fulfilled, and hence \mathcal{B} has a fixed point in \mathfrak{S} . \square

Example 3. Let ρ be a metric on $\mathfrak{S} = \{\Pi|\Pi : [z_1, z_2] \rightarrow [z_1, z_2] \text{ is continuous}\}$ defined by

$$\rho(\Pi_1, \Pi_2) = \sup_{a \in [z_1, z_2]} \{|\Pi_1(a) - \Pi_2(a)|^2\},$$

for all $\Pi_1, \Pi_2 \in \mathfrak{S}$, $0 \leq z_1 < z_2$ such that $\Pi_1 \leq \Pi_2$ and $z_1 \leq \Pi_1(a) \leq \Pi_2(a) \leq z_2$, where $a \in [z_1, z_2]$. A self mapping \mathcal{B} on \mathfrak{S} defined by $\mathcal{B}\Pi = \frac{\Pi}{5}, \Pi \in \mathfrak{S}$ has a unique fixed point with $\tilde{\phi}(\ddot{a}) = \ddot{a}$ and $\tilde{\psi}(\ddot{a}) = \frac{\ddot{a}}{3}$, for all $\ddot{a} \in [0, +\infty]$.

Proof. Since $\min(\Pi_1, \Pi_2)(a) = \min\{\Pi_1(a), \Pi_2(a)\}$ is continuous and all conditions of Corollary 1 are fulfilled for $s = 2$. Hence, we conclude that $0 \in \mathfrak{S}$ is the unique fixed point of \mathcal{B} . \square

3. Application

In this section, as an application of Theorem 3, we will discuss the existence of the unique solution of a nonlinear quadratic integral equation (see [6]).

Let us consider the following nonlinear quadratic integral equation:

$$x(t) = \gamma(t) + \lambda \int_0^1 k_1(t, \omega)g_1(\omega, x(\omega))d\omega \int_0^1 k_2(t, \omega)g_2(\omega, x(\omega))d\omega, \tag{68}$$

$t \in I = [0, 1], \lambda \geq 0$.

Let Γ be a set of all functions $\beta : [0, \infty) \rightarrow [0, \infty)$ such that the following conditions hold:

- (i) β is non-decreasing and $(\beta(t))^q \leq \beta(t^q)$ for all $q \geq 1$.
- (ii) There exist $\phi \in \Phi$ such that $\beta(t) = t - \phi(t)$ for all $t \in [0, \infty)$.

For example, $\beta_1(t) = kt$, where $0 \leq k < 1$ and $\beta_2(t) = \frac{t}{t+1}$ are in Γ [6].

We will study Equation (68) under the following conditions:

- (c₁) $g_i : I \times \mathbb{R} \rightarrow \mathbb{R}, (i = 1, 2)$, where $g_i(t, \theta) \geq 0$ are continuous functions, and there exist two functions $\zeta_i \in L^1(I)$ such that $g_i(t, \theta) \leq \zeta_i(t), (i = 1, 2)$;
- (c₂) $g_1(t, \theta)$ is a monotone non-decreasing in θ and $g_2(t, \zeta)$ is a monotone non-increasing in ζ for all $\theta, \zeta \in \mathbb{R}$ and $t \in I$;
- (c₃) $Y : I \rightarrow \mathbb{R}$ is a continuous function;
- (c₄) $k_i : I \times I \rightarrow \mathbb{R}, (i = 1, 2)$ are continuous in $t \in I$ for every $\omega \in I$ and measurable in $\omega \in I$ for all $t \in I$ such that

$$\int_0^1 k_i(t, \omega)\zeta_i(\omega)d\omega \leq K, i = 1, 2 \text{ and } k_i(t, \theta) \geq 0;$$

- (c₅) there exist constants $0 \leq L_i < 1, (i = 1, 2)$ and $\beta \in \Gamma$ such that for all $\theta, \zeta \in \mathbb{R}$ and $\theta \geq \zeta$,

$$|g_i(t, \theta) - g_i(t, \zeta)| \leq L_i\beta(\theta - \zeta), (i = 1, 2);$$

- (c₆) there exist $m_1, m_2 \in C(I)$ such that

$$\begin{aligned} m_1(t) &\leq \gamma(t) + \lambda \int_0^1 k_1(t, \omega)g_1(\omega, m_1(\omega))d\omega \int_0^1 k_2(t, \omega)g_2(\omega, m_2(\omega))d\omega \\ &\leq \gamma(t) + \lambda \int_0^1 k_1(t, \omega)g_1(\omega, m_2(\omega))d\omega \int_0^1 k_2(t, \omega)g_2(\omega, m_1(\omega))d\omega \\ &\leq m_2(t); \end{aligned}$$

- (c₇) $\max\{L^q, L^q\}\lambda^q K^{2q} \leq \frac{1}{2^{4q-3}}$.

Let $\mathfrak{S} = C(I)$, where $I = [0, 1]$ is the space of continuous functions with the metric

$$d = \sup_{t \in I} |\theta(t) - \zeta(t)|, \text{ for all } \theta, \zeta \in C(I).$$

Then, it is clear that the space can be equipped with a partial order given by

$$\theta, \zeta \in C(I), \theta \leq \zeta \iff \theta(t) \leq \zeta(t), \text{ for all } t \in I.$$

Define a metric ρ for $q \geq 1$ by

$$\rho(\theta, \zeta) = (d(\theta, \zeta))^q = (\sup_{t \in I} |\theta(t) - \zeta(t)|)^q = \sup_{t \in I} |\theta(t) - \zeta(t)|^q, \text{ for all } \theta, \zeta \in C(I).$$

It is obvious that (\mathfrak{S}, ρ) is a complete b -metric space with $s = 2^{q-1}$, Ref. [10].

Furthermore, $\mathfrak{S} \times \mathfrak{S} = C(I) \times C(I)$ is a partially ordered set with the following order relation:

$$\text{for all } (\theta, \zeta), (\rho, \sigma) \in \mathfrak{S} \times \mathfrak{S}, \quad (\theta, \zeta) \leq (\rho, \sigma) \iff \theta \leq \rho \text{ and } \zeta \geq \sigma.$$

Furthermore, for all $\theta, \zeta \in \mathfrak{S}$ and each $t \in I$, $\max\{\theta(t), \zeta(t)\}$ are upper and lower bounds of θ, ζ in \mathfrak{S} . Thus, for every $(\theta, \zeta), (\rho, \sigma) \in \mathfrak{S} \times \mathfrak{S}$, $(\max\{\theta, \rho\}, \min\{\zeta, \sigma\}) \in \mathfrak{S} \times \mathfrak{S}$ is comparable to (θ, ζ) and (ρ, σ) .

Theorem 9. *The integral Equation (68) has a unique solution in $C(I)$ under the hypotheses $(c_1) - (c_7)$.*

Proof. Define a mapping $\mathcal{B} : \mathfrak{S} \times \mathfrak{S} \rightarrow \mathfrak{S}$ by

$$\mathcal{B}(\theta, \zeta)(t) = \gamma(t) + \lambda \int_0^1 k_1(t, \omega)g_1(\omega, \theta(\omega))d\omega \int_0^1 k_2(t, \omega)g_2(\omega, \zeta(\omega))d\omega, \text{ for all } t \in I.$$

Then \mathcal{B} is well defined by the hypotheses. Next, we prove that \mathcal{B} has the mixed monotone property. Consider, for $\theta_1 \leq \theta_2$ and $t \in I$

$$\begin{aligned} &\mathcal{B}(\theta_1, \zeta)(t) - \mathcal{B}(\theta_2, \zeta)(t) = \gamma(t) \\ &+ \lambda \int_0^1 k_1(t, \omega)g_1(\omega, \theta_1(\omega))d\omega \int_0^1 k_2(t, \omega)g_2(\omega, \zeta(\omega))d\omega \\ &- \gamma(t) - \lambda \int_0^1 k_1(t, \omega)g_1(\omega, \theta_2(\omega))d\omega \int_0^1 k_2(t, \omega)g_2(\omega, \zeta(\omega))d\omega \\ &= \lambda \int_0^1 k_1(t, \omega)[g_1(\omega, \theta_1(\omega)) - g_1(\omega, \theta_2(\omega))]d\omega \int_0^1 k_2(t, \omega)g_2(\omega, \zeta(\omega))ds \\ &\leq 0. \end{aligned}$$

Using a similar procedure, we can prove that $\mathcal{B}(\theta, \zeta_1)(t) \leq \mathcal{B}(\theta, \zeta_2)(t)$, if $\zeta_1 \leq \zeta_2$ and $t \in I$. Hence, \mathcal{B} has the mixed monotone property. Moreover, for $(\theta, \zeta) \leq (\rho, \sigma)$, that is, $\theta \leq \rho$ and $\zeta \geq \sigma$, we have

$$\begin{aligned} |\mathcal{B}(\theta, \zeta)(t) - \mathcal{B}(\rho, \sigma)(t)| &\leq |\lambda \int_0^1 k_1(t, \omega)g_1(\omega, \theta(\omega))d\omega \int_0^1 k_2(t, \omega)[g_2(\omega, \zeta(\omega)) \\ &- g_2(\omega, \sigma(\omega))]d\omega \\ &+ \lambda \int_0^1 k_2(t, \omega)g_2(\omega, \sigma(\omega))d\omega \int_0^1 k_1(t, \omega)[g_1(\omega, \theta(\omega)) - g_1(\omega, \rho(\omega))]d\omega| \\ &\leq \lambda \int_0^1 k_1(t, \omega)g_1(\omega, \theta(\omega))d\omega \int_0^1 k_2(t, \omega)|g_2(\omega, \zeta(\omega)) - g_2(\omega, \sigma(\omega))|d\omega \\ &+ \lambda \int_0^1 k_2(t, \omega)g_2(\omega, \sigma(\omega))d\omega \int_0^1 k_1(t, \omega)|g_1(\omega, \theta(\omega)) - g_1(\omega, \rho(\omega))|d\omega \\ &\leq \lambda \int_0^1 k_1(t, \omega)m_1(\omega)d\omega \int_0^1 k_2(t, \omega)L_2\beta[|\zeta(\omega) - \sigma(\omega)|]d\omega \\ &+ \lambda \int_0^1 k_2(t, \omega)m_2(\omega)d\omega \int_0^1 k_1(t, \omega)L_1\beta[|\rho(\omega) - \theta(\omega)|]d\omega. \end{aligned}$$

Since the function β is non-decreasing and, $\theta \leq \rho$ and $\zeta \geq \sigma$, we have

$$\beta(\rho(\omega) - \theta(\omega)) \leq \beta(\sup_{t \in I} |\theta(\omega) - \rho(\omega)|) = \beta(d(\theta, \rho))$$

and

$$\beta(\zeta(\omega) - \sigma(\omega)) \leq \beta(\sup_{t \in I} |\zeta(\omega) - \sigma(\omega)|) = \beta(d(\zeta, \sigma)).$$

Thus,

$$\begin{aligned}
 |\mathcal{B}(\theta, \zeta(t)) - \mathcal{B}(\rho, \sigma)(t)| &\leq \lambda K \int_0^1 k_2(t, \omega) L_2 \beta(d(\zeta, \sigma)) d\omega \\
 &\quad + \lambda K \int_0^1 k_1(t, \omega) L_1 \beta(d(\rho, \theta)) d\omega \\
 &\leq \lambda K^2 \max\{L_1, L_2\} [\beta(d(\rho, \theta)) + \beta(d(\zeta, \sigma))].
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \mathcal{J}(\mathcal{B}(\theta, \zeta), \mathcal{B}(\rho, \sigma)) &= \sup_{t \in I} |\mathcal{B}(\theta, \zeta)(t) - \mathcal{B}(\rho, \sigma)(t)|^q \\
 &\leq \{\lambda K^2 \max\{L_1, L_2\} [\beta(d(\rho, \theta)) + \beta(d(\zeta, \sigma))]\}^q \\
 &= \lambda^q K^{2q} \max\{L_1^q, L_2^q\} [\beta(d(\rho, \theta)) + \beta(d(\zeta, \sigma))]^q,
 \end{aligned}$$

and from the fact that $(a + b)^q \leq 2^{q-1}(a^q + b^q)$, for all $a, b \in (0, \infty)$ and $q > 1$, we have

$$\begin{aligned}
 \mathcal{J}(\mathcal{B}(\theta, \zeta), \mathcal{B}(\rho, \sigma)) &\leq 2^{q-1} \lambda^q K^{2q} \max\{L_1^q, L_2^q\} [(\beta(d(\rho, \theta)))^q + (\beta(d(\zeta, \sigma)))^q] \\
 &\leq 2^{q-1} \lambda^q K^{2q} \max\{L_1^q, L_2^q\} [\beta(d(\rho, \theta)) + \beta(d(\zeta, \sigma))]^q \\
 &\leq 2^q \lambda^q K^{2q} \max\{L_1^q, L_2^q\} [\beta \mathcal{E}_\ell(\theta, \zeta, \rho, \sigma)] \\
 &\leq 2^q \lambda^q K^{2q} \max\{L_1^q, L_2^q\} [\mathcal{E}_\ell(\theta, \zeta, \rho, \sigma) - \tilde{\Psi}(\mathcal{E}_\ell(\theta, \zeta, \rho, \sigma))] \\
 &\leq \frac{1}{2^{3q-k}} \mathcal{E}_\ell(\theta, \zeta, \rho, \sigma) - \frac{1}{2^{3q-k}} \tilde{\Psi}(\mathcal{E}_\ell(\theta, \zeta, \rho, \sigma)).
 \end{aligned}$$

which implies that the mapping \mathcal{B} satisfies the contractive condition (54) appearing in Corollary 3.

Finally, let m_1, m_2 be the functions appearing in assumption (c_6) ; then, by (c_6) , we obtain

$$m_1 \leq \mathcal{B}(m_1, m_2) \leq \mathcal{B}(m_2, m_1) \leq m_2.$$

Therefore from Theorem 7, \mathcal{B} has a unique coupled fixed point $(\mathcal{J}^*, \mathcal{H}^*) \in \mathfrak{S} \times \mathfrak{S}$. Since $m_1 \leq m_2$, then from Theorem 8, $\mathcal{J}^* = \mathcal{H}^*$ which suggests that $\mathcal{J}^* = \mathcal{B}(\mathcal{J}^*, \mathcal{J}^*)$. Therefore, $\mathcal{J}^* \in C(I)$ is the unique solution of Equation (68). \square

4. Conclusions

In this work, we introduced generalized weak contractions involving the altering distance functions in which conditions appear in the form of a fraction. The results obtained in this paper are generalizing and extending the results of [22] and many comparable results in the literature. Further, a few examples are given to justify the findings.

Recently, George et. al. [27] have introduced rectangular b -metric spaces. Furthermore, Mitrović and Radenović [28] introduced $b_v(s)$ -metric space. It is an interesting opening problem to study generalized weak contractions having the altering distance functions in those spaces.

In conclusion, we provide an open question. Can we replace condition (1) with a weaker condition

$$\tilde{\phi}(\mathcal{J}(T\theta, T\zeta)) \leq \tilde{\phi}(\mathcal{D}(\theta, \zeta)) - \tilde{\Psi}(\mathcal{D}(\theta, \zeta)) + \Lambda \Omega(\mathcal{E}(\theta, \zeta))? \tag{69}$$

Furthermore, the above results can be generalized and extended by introducing the concept of w t -distance on a metric-type space [29], cone b -metric spaces over Banach algebra [30] and (ϕ, ψ) -weak contractions [31].

Author Contributions: Conceptualization, N.S.R. and Z.D.M.; formal analysis, Z.D.M., D.S. and N.M.; writing—original draft preparation, N.S.R., D.S., N.M. and Z.D.M.; writing—review and editing, N.S.R., Z.D.M., D.S. and N.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors N. Mlaiki and D. Santina would like to thank Prince Sultan University for paying the publication fees for this work through TAS LAB.

Conflicts of Interest: The authors declare no conflict of interest.

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