Heliyon 8 (2022) e12442

Contents lists available at ScienceDirect

Heliyon

journal homepage: www.cell.com/heliyon

Research article

Coupled fixed points of $(\widehat{\phi}, \widehat{\psi}, \widehat{\theta})$-contractive mappings in partially ordered b-metric spaces

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ARTICLE INFO

Keywords:
Fixed point
Compatible and weakly compatible mappings
Coincidence and coupled coincidence points
$(\widehat{\phi}, \widehat{\psi}, \widehat{\theta})$-contraction
Uniqueness fixed points

ABSTRACT

We aim to prove the existence and uniqueness of the fixed points for the self mappings satisfying generalized contractions involving altering distance functions in ordered metric type space. The results obtained in this work are generalizing some important findings in the literature and few illustrations are given to support the outcomes.

1. Introduction

Metric type space or b-metric space is one of the most important generalization of a usual metric space. It has many applications in scientific and mathematical research. Bakhtin [13], Czerwik [18] have discussed fixed point results over a metric type space very first. In recent times more works have been done that focus on to acquire fixed points and, then further extended for coincidence, coupled coincidence points of the maps satisfies various contraction conditions in this context. In connection they explored several applications of differential and integral equations, the readers may refer from [15, 17, 20, 22, 27, 30, 31, 32, 38, 39, 40] and from the references provided therein. In addition, by implementing necessary topological properties on a space and/or mappings which are either single or multi valued in ordered metric type space, several authors have been generalized and extended the results, some of such are from [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16, 18, 19, 21, 23, 24, 25, 26, 28, 29, 33, 34, 35, 36, 37, 43], which create a natural interest in this direction. Very recently, Seshagiri Rao et al. [41, 42] investigated fixed point results in ordered metric type spaces for the mappings satisfying generalized weak contractions involving altering distance functions. Also one can see some important generalizations of the fixed points results in Gb-metric space and an extended fuzzy cone b-metric space by Vishal Gupta et al. [44, 45] which will enhance the results obtained in this work.

Now, in this work for obtaining the fixed point of a map $\mathcal{J} : \mathcal{Q} \to \mathcal{Q}$, we initiate a generalized contraction given below which is involving altering distance functions $\widehat{\phi} \in \widehat{\Phi}$, $\widehat{\psi} \in \widehat{\Psi}$ and $\widehat{\theta} \in \widehat{\Theta}$ defined in $[0, +\infty)$.

$$\widehat{\phi}(\mathscr{s}\Omega(\mathcal{J}\mathscr{p}, \mathcal{J}\varpi)) \leq \widehat{\phi}(C(\mathscr{p}, \varpi)) - \widehat{\psi}(C(\mathscr{p}, \varpi)) + \mathscr{M}\widehat{\theta}(D(\mathscr{p}, \varpi)), \tag{1}$$

where

$$C(\mathscr{p}, \varpi) = \max\{\frac{\Omega(\varpi, \mathcal{J}\varpi)\left[1 + \Omega(\mathscr{p}, \mathcal{J}\mathscr{p})\right]}{1 + \Omega(\mathscr{p}, \varpi)}, \frac{\Omega(\mathscr{p}, \mathcal{J}\mathscr{p})\,\Omega(\mathscr{p}, \mathcal{J}\varpi)}{1 + \Omega(\mathscr{p}, \mathcal{J}\varpi) + \Omega(\varpi, \mathcal{J}\mathscr{p})}, \Omega(\mathscr{p}, \varpi)\}, \tag{2}$$

and

$$D(\mathscr{p}, \varpi) = \min\{\Omega(\mathscr{p}, \mathcal{J}\mathscr{p}), \Omega(\varpi, \mathcal{J}\varpi), \Omega(\varpi, \mathcal{J}\mathscr{p}), \Omega(\mathscr{p}, \mathcal{J}\varpi)\}, \tag{3}$$

for all $\mathscr{p}, \varpi \in \mathcal{Q}$ such that $\mathscr{p} \preceq \varpi$, $\mathscr{M} \geq 0$, $\mathscr{s} > 1$ and, $(\mathcal{Q}, \Omega, \preceq)$ is a complete partially ordered b-metric space (c.p.o.b-m.s.).

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https://doi.org/10.1016/j.heliyon.2022.e12442
Received 5 February 2022; Received in revised form 1 June 2022; Accepted 9 December 2022

Later, we extend the same contraction condition for a pair mappings in the same context to acquire coincidence, couple coincidence points and a common fixed point. These results are the generalizations of [14, 15, 22, 27, 31] and some other results in the literature. The readers may refer the necessary definitions, properties and lemmas for the present study from the works of [2, 23, 31, 36, 41, 42].

We employ the following distance functions defined in $[0,+\infty)$ all over the work in this paper.

(a). A mapping $\hat{\phi}:[0,+\infty)\to[0,+\infty)$ is continuously non-decreasing and $\hat{\phi}(\eta)=0$ iff $\eta=0$, for any $\eta\in[0,+\infty)$. Denote all such kind of functions by $\hat{\Phi}$.
(b). A mapping $\hat{\psi}:[0,+\infty)\to[0,+\infty)$ is such that $\hat{\psi}$ is lower semi-continuous and $\hat{\psi}(\eta)=0$ iff $\eta=0$. Signify all of such functions by $\hat{\Psi}$.
(c). A mapping $\hat{\theta}:[0,+\infty)\to[0,+\infty)$ is continuous and $\hat{\theta}(\eta)=0$ iff $\eta=0$. Designate all such functions by Θ.

2. Main results

This section starts with the following theorem in a metric type space.

Theorem 2.1. *A non-decreasing continuous self mapping $\mathcal{J}$ defined on a c.p.o.b-m.s. $\mathcal{Q}$ with regards to $\preceq$ has a fixed point by satisfying contraction (1) and for certain $p_0\in\mathcal{Q}$ such that $p_0\preceq\mathcal{J}p_0$.*

Proof. If certain $p_0\in\mathcal{Q}$ with $\mathcal{J}p_0=p_0$ then is our result. If not then $p_0\prec\mathcal{J}p_0$ and then define a sequence $\{p_n\}$ in $\mathcal{Q}$ such that $p_{n+1}=\mathcal{J}p_n$ $(n\geq 0)$. As from the property of $\mathcal{J}$, we get

$$p_0\prec\mathcal{J}p_0=p_1\preceq\ldots\preceq p_n\preceq\mathcal{J}p_n=p_{n+1}\preceq\ldots\,. \tag{4}$$

If $p_{n_0}=p_{n_0+1}$ for certain $n_0\in\mathbb{N}$, then $p_{n_0}\in\mathcal{Q}$ is a fixed point of $\mathcal{J}$ from equation (4). In contrary, for all n, $p_n\neq p_{n+1}$ and then $p_n\succ p_{n-1},(n\geq 1)$ by definition. As from equation (1), we get

$$\begin{aligned}\hat{\phi}(\Omega(p_n,p_{n+1}))&=\hat{\phi}(\Omega(\mathcal{J}p_{n-1},\mathcal{J}p_n))\leq\hat{\phi}(s\Omega(\mathcal{J}p_{n-1},\mathcal{J}p_n))\\&\leq\hat{\phi}(C(p_{n-1},p_n))-\hat{\psi}(C(p_{n-1},p_n))+\mathcal{M}\hat{\theta}(D(p_{n-1},p_n)),\end{aligned}\tag{5}$$

where

$$\begin{aligned}C(p_{n-1},p_n)=\max\{&\frac{\Omega(p_n,\mathcal{J}p_n)\left[1+\Omega(p_{n-1},\mathcal{J}p_{n-1})\right]}{1+\Omega(p_{n-1},p_n)},\frac{\Omega(p_{n-1},\mathcal{J}p_{n-1})\,\Omega(p_{n-1},\mathcal{J}p_n)}{1+\Omega(p_{n-1},\mathcal{J}p_n)+\Omega(p_n,\mathcal{J}p_{n-1})},\\&\Omega(p_{n-1},p_n)\},\\=\max\{&\Omega(p_n,p_{n+1}),\frac{\Omega(p_{n-1},p_n)\,\Omega(p_{n-1},p_{n+1})}{1+\Omega(p_{n-1},p_{n+1})+\Omega(p_n,p_n)},\ \Omega(p_{n-1},p_n)\}\\\leq\max\{&\Omega(p_n,p_{n+1}),\Omega(p_{n-1},p_n)\},\end{aligned}\tag{6}$$

and

$$D(p_{n-1},p_n)=\min\{\Omega(p_{n-1},\mathcal{J}p_{n-1}),\Omega(p_n,\mathcal{J}p_n),\Omega(p_n,\mathcal{J}p_{n-1}),\Omega(p_{n-1},\mathcal{J}p_n)\}=0. \tag{7}$$

From equations (5), (6) and (7), we have

$$\Omega(p_n,p_{n+1})=\Omega(\mathcal{J}p_{n-1},\mathcal{J}p_n)\leq\frac{1}{s}C(p_{n-1},p_n). \tag{8}$$

Suppose $\max\{\Omega(p_n,p_{n+1}),\Omega(p_{n-1},p_n)\}=\Omega(p_n,p_{n+1})$ for certain $n\geq 1$, equation (8) implies that

$$\Omega(p_n,p_{n+1})\leq\frac{1}{s}\Omega(p_n,p_{n+1}), \tag{9}$$

which leads to a contradiction in equation (9). Thus, $\max\{\Omega(p_n,p_{n+1}),\Omega(p_{n-1},p_n)\}=\Omega(p_{n-1},p_n),(n\geq 1)$ and hence the equation (8) becomes

$$\Omega(p_n,p_{n+1})\leq\frac{1}{s}\Omega(p_{n-1},p_n). \tag{10}$$

Since $0<\frac{1}{s}<1$ and the results from [1, 6, 12, 21] suggests that $\{p_n\}$ is a convergent Cauchy sequence in $\mathcal{Q}$, as $\mathcal{Q}$ is complete. Therefore, $p_n\to\eta\in\mathcal{Q}$.

In addition from the continuity property of $\mathcal{J}$, we have

$$\mathcal{J}\eta=\mathcal{J}(\lim_{n\to+\infty}p_n)=\lim_{n\to+\infty}\mathcal{J}p_n=\lim_{n\to+\infty}p_{n+1}=\eta, \tag{11}$$

this exhibits in equation (11) that η is a fixed point of $\mathcal{J}$. □

The mapping $\mathcal{J}$ need not be continuous in Theorem 2.1 as a result, we have the following theorem by implementing the condition below on $\mathcal{Q}$:

$$\begin{aligned}&\text{Let }\{p_n\}\text{ be a non-decreasing sequence in }\mathcal{Q}\text{ with }p_n\to\eta,\text{ for some }\eta\in\mathcal{Q}\text{ then}\\&p_n\preceq\eta\ ,(n\in\mathbb{N}),\ \text{i.e., }\ \eta=\sup p_n.\end{aligned}\tag{12}$$

Theorem 2.2. *If condition (12) holds by $\mathcal{Q}$ in Theorem 2.1, then $\mathcal{J}$ has a fixed point.*

Proof. By Theorem 2.1, there is a Cauchy sequence $\{p_n\}$ in $\mathcal{Q}$ so that $p_n\to\eta$, for some $\eta\in\mathcal{Q}$. Also, as a result from (12), we have $p_n\preceq\eta,(n\geq 0)$, i.e. $\eta=\sup p_n$.

Next, to show that $\mathcal{J}\eta = \eta$. In contrary, $\mathcal{J}\eta \neq \eta$, then

$$C(p_n, \eta) = \max\{\frac{\Omega(\eta, \mathcal{J}\eta)\left[1 + \Omega(p_n, \mathcal{J}p_n)\right]}{1 + \Omega(p_n, \eta)}, \frac{\Omega(p_n, \mathcal{J}p_n)\,\Omega(p_n, \mathcal{J}\eta)}{1 + \Omega(p_n, \mathcal{J}\eta) + \Omega(\eta, \mathcal{J}p_n)}, \Omega(p_n, \eta)\}, \tag{13}$$

and

$$\mathcal{D}(p_n, \eta) = \min\{\Omega(p_n, \mathcal{J}p_n), \Omega(\eta, \mathcal{J}\eta), \Omega(\eta, \mathcal{J}p_n), \Omega(p_n, \mathcal{J}\eta)\}. \tag{14}$$

Taking $n \to +\infty$ in equations (13) and (14) and also from $\lim_{n\to+\infty} p_n = \eta$, we get

$$\lim_{n\to+\infty} C(p_n, \eta) = \max\{\Omega(\eta, \mathcal{J}\eta), 0\} = \Omega(\eta, \mathcal{J}\eta) \tag{15}$$

and

$$\lim_{n\to+\infty} \mathcal{D}(p_n, \eta) = \min\{\Omega(\eta, \mathcal{J}\eta), 0\} = 0. \tag{16}$$

We know that, $p_n \preceq \eta, \forall n$, and thus (1) becomes

$$\begin{aligned}\hat{\phi}(\Omega(p_{n+1}, \mathcal{J}\eta)) = \hat{\phi}(\Omega(\mathcal{J}p_n, \mathcal{J}\eta)) &\le \hat{\phi}(s\Omega(\mathcal{J}p_n, \mathcal{J}\eta)) \\ &\le \hat{\phi}(C(p_n, \eta)) - \hat{\psi}(C(p_n, \eta)) + \mathcal{M}\hat{\theta}(\mathcal{D}(p_n, \eta)).\end{aligned} \tag{17}$$

Taking $n \to +\infty$ in equation (17) and also from the equations (15) and (16), we obtain that

$$\hat{\phi}(\Omega(\eta, \mathcal{J}\eta)) \le \hat{\phi}(\Omega(\eta, \mathcal{J}\eta)) - \hat{\psi}(\Omega(\eta, \mathcal{J}\eta)) < \hat{\phi}(\Omega(\eta, \mathcal{J}\eta)), \tag{18}$$

this is a contradiction from (18). Hence, $\mathcal{J}\eta = \eta$. □

Theorem 2.3. *The mapping $\mathcal{J}$ in Theorems 2.1 & 2.2 has a unique fixed point, if $\mathcal{Q}$ is comparable.*

Proof. Suppose that p^*, ϖ^* be any two distinct fixed points of $\mathcal{J}$, thus from (1)

$$\begin{aligned}\hat{\phi}(\Omega(\mathcal{J}p^*, \mathcal{J}\varpi^*)) &\le \hat{\phi}(s\Omega(\mathcal{J}p^*, \mathcal{J}\varpi^*)) \\ &\le \hat{\phi}(C(p^*, \varpi^*)) - \hat{\psi}(C(p^*, \varpi^*)) + \mathcal{M}\hat{\theta}(\mathcal{D}(p^*, \varpi^*)),\end{aligned} \tag{19}$$

where

$$\begin{aligned}C(p^*, \varpi^*) &= \max\{\frac{\Omega(\varpi^*, \mathcal{J}\varpi^*)\left[1 + \Omega(p^*, \mathcal{J}p^*)\right]}{1 + \Omega(p^*, \varpi^*)}, \frac{\Omega(p^*, \mathcal{J}p^*)\,\Omega(p^*, \mathcal{J}\varpi^*)}{1 + \Omega(p^*, \mathcal{J}\varpi^*) + \Omega(\varpi^*, \mathcal{J}p^*)}, \Omega(p^*, \varpi^*)\} \\ &= \max\{\frac{\Omega(\varpi^*, \varpi^*)\left[1 + \Omega(p^*, p^*)\right]}{1 + \Omega(p^*, \varpi^*)}, \frac{\Omega(p^*, p^*)\,\Omega(p^*, \varpi^*)}{1 + \Omega(p^*, \varpi^*) + \Omega(p^*, \varpi^*)}, \Omega(p^*, \varpi^*)\} \\ &= \max\{0, \Omega(p^*, \varpi^*)\} \\ &= \Omega(p^*, \varpi^*)\end{aligned} \tag{20}$$

and

$$\mathcal{D}(p^*, \varpi^*) = \min\{\Omega(p^*, \mathcal{J}p^*), \Omega(\varpi^*, \mathcal{J}\varpi^*), \Omega(\varpi^*, \mathcal{J}p^*), \Omega(p^*, \mathcal{J}\varpi^*)\} = 0. \tag{21}$$

Equation (21) implies that

$$\Omega(p^*, \varpi^*) = \Omega(\mathcal{J}p^*, \mathcal{J}\varpi^*) \le \frac{1}{s} C(p^*, \varpi^*), \tag{22}$$

and hence from (22), we get

$$\Omega(p^*, \varpi^*) \le \frac{1}{s}\Omega(p^*, \varpi^*) < \Omega(p^*, \varpi^*), \tag{23}$$

which leads a contradiction to $p^* \neq \varpi^*$ in (23). Therefore, $p^* = \varpi^*$. □

We can get the below consequence from Theorems 2.1, 2.2 & 2.3.

Corollary 2.4. *The same conclusions will be obtained as from Theorems 2.1, 2.2 & 2.3 by putting $\mathcal{M} = 0$ in (1).*

Corollary 2.5. *By replacing $\hat{\phi}(r) = r$ and $\hat{\psi}(r) = (1 - \ell)r$ in Corollary 2.4, then the similar conclusions of Theorems 2.1-2.3 will be acquired with the following contraction condition*

$$\Omega(\mathcal{J}p, \mathcal{J}\varpi) \le \frac{\ell}{s} \max\{\frac{\Omega(\varpi, \mathcal{J}\varpi)\left[1 + \Omega(p, \mathcal{J}p)\right]}{1 + \Omega(p, \varpi)}, \frac{\Omega(p, \mathcal{J}p)\,\Omega(p, \mathcal{J}\varpi)}{1 + \Omega(p, \mathcal{J}\varpi) + \Omega(\varpi, \mathcal{J}p)}, \Omega(p, \varpi)\}. \tag{24}$$

Definition 2.6. A generalized contraction of a self-map $\mathcal{J}$ on $\mathcal{Q}$ with regards to a mapping $g : \mathcal{Q} \to \mathcal{Q}$ is defined by

$$\hat{\phi}(\mathfrak{s}\Omega(\mathcal{J}\wp, \mathcal{J}\varpi)) \leq \hat{\phi}(C_{g}(\wp, \varpi)) - \hat{\psi}(C_{g}(\wp, \varpi)) + \mathcal{M}\hat{\theta}(D_{g}(\wp, \varpi)), \tag{25}$$

where

$$C_{g}(g\wp, g\varpi) = \max\{\frac{\Omega(g\varpi, \mathcal{J}\varpi)\left[1+\Omega(g\wp, \mathcal{J}\wp)\right]}{1+\Omega(g\wp, g\varpi)}, \frac{\Omega(g\wp, \mathcal{J}\wp)\,\Omega(g\wp, \mathcal{J}\varpi)}{1+\Omega(g\wp, \mathcal{J}\varpi)+\Omega(g\varpi, \mathcal{J}\wp)}, \Omega(g\wp, g\varpi)\}, \tag{26}$$

and

$$D_{g}(g\wp, g\varpi) = \min\{\Omega(g\wp, \mathcal{J}\wp), \Omega(g\varpi, \mathcal{J}\varpi), \Omega(g\varpi, \mathcal{J}\wp), \Omega(g\wp, \mathcal{J}\varpi)\}, \tag{27}$$

for all $\wp, \varpi \in \mathcal{Q}$ with $g\wp \preceq g\varpi$, $\hat{\phi} \in \hat{\Phi}$, $\hat{\psi} \in \hat{\Psi}$ and $\hat{\theta} \in \hat{\Theta}$.

Theorem 2.7. *A coincidence point exists for the continuous mappings $\mathcal{J}$ and g defined in above Definition 2.6 satisfies condition (25) with the following assumptions:*

(a). $\mathcal{J}$ is monotone g-non-decreasing mapping,
(b). $\mathcal{J}\mathcal{Q} \subseteq g\mathcal{Q}$,
(c). $\mathcal{J}$ and g are compatible mappings,
(c). $g\wp_0 \preceq \mathcal{J}\wp_0$ for certain $\wp_0 \in \mathcal{Q}$ and
(d). $(\mathcal{Q}, \Omega, \preceq)$ is complete.

Proof. By Theorem 2.2 [8], there exists two sequences $\{\wp_n\}, \{\varpi_n\} \subseteq \mathcal{Q}$ such that

$$\varpi_n = \mathcal{J}\wp_n = g\wp_{n+1} \text{ for all } n \geq 0, \tag{28}$$

for which

$$g\wp_0 \preceq g\wp_1 \preceq .... \preceq g\wp_n \preceq g\wp_{n+1} \preceq .... . \tag{29}$$

Now from [8], we have to show that

$$\Omega(\varpi_n, \varpi_{n+1}) \leq \lambda\Omega(\varpi_{n-1}, \varpi_n) \ (n \geq 1), \tag{30}$$

where $0 \leq \lambda < \frac{1}{\mathfrak{s}}$. From the equations (25)-(29), we have

$$\begin{aligned}\hat{\phi}(\mathfrak{s}\Omega(\varpi_n, \varpi_{n+1})) &= \hat{\phi}(\mathfrak{s}\Omega(\mathcal{J}\wp_n, \mathcal{J}\wp_{n+1}))\\ &\leq \hat{\phi}(C_{g}(\wp_n, \wp_{n+1})) - \hat{\psi}(C_{g}(\wp_n, \wp_{n+1})) + \mathcal{M}\hat{\theta}(D_{g}(\wp_n, \wp_{n+1})),\end{aligned} \tag{31}$$

where

$$\begin{aligned}C_{g}(\wp_n, \wp_{n+1}) &= \max\{\frac{\Omega(g\wp_{n+1}, \mathcal{J}\wp_{n+1})\left[1+\Omega(g\wp_n, \mathcal{J}\wp_n)\right]}{1+\Omega(g\wp_n, g\wp_{n+1})},\\ &\qquad \frac{\Omega(g\wp_n, \mathcal{J}\wp_n)\,\Omega(g\wp_n, \mathcal{J}\wp_{n+1})}{1+\Omega(g\wp_n, \mathcal{J}\wp_{n+1})+\Omega(g\wp_{n+1}, \mathcal{J}\wp_n)}, \Omega(g\wp_n, g\wp_{n+1})\}\\ &= \max\{\frac{\Omega(\varpi_n, \varpi_{n+1})\left[1+\Omega(\varpi_{n-1}, \varpi_n)\right]}{1+\Omega(\varpi_{n-1}, \varpi_n)},\\ &\qquad \frac{\Omega(\varpi_{n-1}, \varpi_n)\,\Omega(\varpi_{n-1}, \varpi_{n+1})}{1+\Omega(\varpi_{n-1}, \varpi_{n+1})+\Omega(\varpi_n, \varpi_n)}, \Omega(\varpi_{n-1}, \varpi_n)\}\\ &\leq \max\{\Omega(\varpi_{n-1}, \varpi_n), \Omega(\varpi_n, \varpi_{n+1})\}\end{aligned} \tag{32}$$

and

$$\begin{aligned}D_{g}(\wp_n, \wp_{n+1}) &= \min\{\Omega(g\wp_n, \mathcal{J}\wp_n), \Omega(g\wp_{n+1}, \mathcal{J}\wp_{n+1}), \Omega(g\wp_{n+1}, \mathcal{J}\wp_n), \Omega(g\wp_n, \mathcal{J}\wp_{n+1})\}\\ &= \min\{\Omega(\varpi_{n-1}, \varpi_n), \Omega(\varpi_n, \varpi_{n+1}), \Omega(\varpi_n, \varpi_n), \Omega(\varpi_{n-1}, \varpi_{n+1})\} = 0.\end{aligned} \tag{33}$$

From (31), we get

$$\hat{\phi}(\mathfrak{s}\Omega(\varpi_n, \varpi_{n+1})) \leq \hat{\phi}(\max\{\Omega(\varpi_{n-1}, \varpi_n), \Omega(\varpi_n, \varpi_{n+1})\}) - \hat{\psi}(\max\{\Omega(\varpi_{n-1}, \varpi_n), \Omega(\varpi_n, \varpi_{n+1})\}). \tag{34}$$

If $0 < \Omega(\varpi_{n-1}, \varpi_n) \leq \Omega(\varpi_n, \varpi_{n+1})$ for certain n, then equation (34) follows that

$$\hat{\phi}(\mathfrak{s}\Omega(\varpi_n, \varpi_{n+1})) \leq \hat{\phi}(\Omega(\varpi_n, \varpi_{n+1})) - \hat{\psi}(\Omega(\varpi_n, \varpi_{n+1})) < \hat{\phi}(\Omega(\varpi_n, \varpi_{n+1})), \tag{35}$$

or equivalently

$$\mathfrak{s}\Omega(\varpi_n, \varpi_{n+1}) \leq \Omega(\varpi_n, \varpi_{n+1}), \tag{36}$$

which is a contradiction. Therefore, the equation (34) becomes

$$\mathfrak{s}\Omega(\varpi_n, \varpi_{n+1}) \leq \Omega(\varpi_{n-1}, \varpi_n). \tag{37}$$

Hence $0 \leq \lambda < \frac{1}{\mathfrak{s}}$ from (30). According to Lemma 3.1 [28] and, from equation (30), we get

$$\lim_{n\to+\infty} \mathcal{J}\wp_n = \lim_{n\to+\infty} g\wp_{n+1} = \mu, \text{ where } \mu \in \mathcal{Q}. \tag{38}$$

Also by the condition (c), we get

$$\lim_{n\to+\infty} \Omega(g(\mathcal{J}\wp_n), \mathcal{J}(g\wp_n)) = 0. \tag{39}$$

Moreover, by continuity of $\mathcal{J}$, g we have

$$\lim_{n\to+\infty} g(\mathcal{J}\wp_n) = g\mu, \qquad \lim_{n\to+\infty} \mathcal{J}(g\wp_n) = \mathcal{J}\mu. \tag{40}$$

Furthermore,

$$\frac{1}{s}\Omega(\mathcal{J}\mu, g\mu) \le \Omega(\mathcal{J}\mu, \mathcal{J}(g\wp_n)) + s\Omega(\mathcal{J}(g\wp_n), g(\mathcal{J}\wp_n)) + s\Omega(g(\mathcal{J}\wp_n), g\mu). \tag{41}$$

Therefore, $\Omega(\mathcal{J}v, gv) = 0$ as $n \to +\infty$ in (41) and from (38)-(40). Hence the proof. □

We have the accompanying result without the continuity of g, $\mathcal{J}$ in hypotheses of Theorem 2.7 and $\mathcal{Q}$ has the below property:

$$\begin{aligned}&\text{A sequence } \{g\wp_n\} \in \mathcal{Q}, \text{ which is nondecreasing with } \lim_{n\to+\infty} g\wp_n = g\wp \in g\mathcal{Q}, \text{where } g\mathcal{Q}\\ &\text{is a closed subset of } \mathcal{Q} \text{ and } g\wp_n \preceq g\wp, g\wp \preceq g(g\wp) \text{ for } n \text{ with } g\wp_0 \preceq \mathcal{J}\wp_0 \text{ for some } \wp_0 \in \mathcal{Q}.\end{aligned} \tag{42}$$

Theorem 2.8. *If $\mathcal{Q}$ satisfies condition (42) in Theorem 2.7, then*

(a). $\mathcal{J}$ and g have a coincidence point when $\mathcal{J}$ and g are weakly compatible and
(b). $\mathcal{J}$, g have a common fixed point, when $\mathcal{J}$ and g are commuting at their coincidence points.

Proof. Since $\{\varpi_n\} = \{\mathcal{J}\wp_n\} = \{g\wp_{n+1}\}$ is a Cauchy sequence by Theorem 2.7 and hence by completeness of $g\mathcal{Q}$, $\lim_{n\to+\infty} \mathcal{J}\wp_n = \lim_{n\to+\infty} g\wp_{n+1} \to g\mu$, some $\mu \in \mathcal{Q}$ as $g\mathcal{Q}$ is closed. Also, $g\wp_n \preceq g\mu, (n \ge 0)$, we obtain that

$$\hat{\phi}(s\Omega(\mathcal{J}\wp_n, \mathcal{J}\mu)) \le \hat{\phi}(C_g(\wp_n, \mu)) - \hat{\psi}(C_g(\wp_n, \mu)) + \mathcal{M}\hat{\theta}(\mathcal{D}_g(\wp_n, \mu)), \tag{43}$$

where

$$\begin{aligned} C_g(\wp_n, \mu) &= \max\{\frac{\Omega(g\mu, \mathcal{J}\mu)\left[1 + \Omega(g\wp_n, \mathcal{J}\wp_n)\right]}{1 + \Omega(g\wp_n, g\mu)}, \frac{\Omega(g\wp_n, \mathcal{J}\wp_n)\,\Omega(g\wp_n, \mathcal{J}\mu)}{1 + \Omega(g\wp_n, \mathcal{J}\mu) + \Omega(g\mu, \mathcal{J}\wp_n)}, \Omega(g\wp_n, g\mu)\}\\ &\to \max\{\Omega(g\mu, \mathcal{J}\mu), 0\}\\ &= \Omega(g\mu, \mathcal{J}\mu) \text{ as } n \to +\infty, \end{aligned} \tag{44}$$

and,

$$\begin{aligned} \mathcal{D}_g(\wp_n, \mu) &= \min\{\Omega(g\wp_n, \mathcal{J}\wp_n), \Omega(g\mu, \mathcal{J}\mu), \Omega(g\mu, \mathcal{J}\wp_n), \Omega(g\wp_n, \mathcal{J}\mu)\}\\ &\to \min\{\Omega(g\mu, \mathcal{J}\mu), 0\}\\ &= 0 \text{ as } n \to +\infty. \end{aligned} \tag{45}$$

As a consequence, the equation (43) suggests that,

$$\hat{\phi}(s\lim_{n\to+\infty} \Omega(\mathcal{J}\wp_n, \mathcal{J}\mu)) \le \hat{\phi}(\Omega(g\mu, \mathcal{J}\mu)) - \hat{\psi}(\Omega(g\mu, \mathcal{J}\mu)) < \hat{\phi}(\Omega(g\mu, \mathcal{J}\mu)). \tag{46}$$

Thus,

$$\lim_{n\to+\infty} \Omega(\mathcal{J}\wp_n, \mathcal{J}\mu) < \frac{1}{s}\Omega(g\mu, \mathcal{J}\mu). \tag{47}$$

Furthermore, the metric triangular inequality follows that

$$\frac{1}{s}\Omega(g\mu, \mathcal{J}\mu) \le \Omega(g\mu, \mathcal{J}\wp_n) + \Omega(\mathcal{J}\wp_n, \mathcal{J}\mu). \tag{48}$$

If $g\mu \ne \mathcal{J}\mu$ then (47) and (48) lead to a contradiction. Therefore, $g\mu = \mathcal{J}\mu$. Suppose that, $g\mu = \mathcal{J}\mu = \rho$, then $\mathcal{J}\rho = \mathcal{J}(g\mu) = g(\mathcal{J}\mu) = g\rho$. Again from (43) with $g\mu = g(g\mu) = g\rho$ and $g\mu = \mathcal{J}\mu$, $g\rho = \mathcal{J}\rho$, we get

$$\hat{\phi}(s\Omega(\mathcal{J}\mu, \mathcal{J}\rho)) \le \hat{\phi}(C_g(\mu, \rho)) - \hat{\psi}(C_g(\mu, \rho)) < \hat{\phi}(\Omega(\mathcal{J}\mu, \mathcal{J}\rho)), \tag{49}$$

or equivalently,

$$s\Omega(\mathcal{J}\mu, \mathcal{J}\rho) \le \Omega(\mathcal{J}\mu, \mathcal{J}\rho). \tag{50}$$

If $\mathcal{J}\mu \ne \mathcal{J}\rho$, then from (50) follows a contradiction. Therefore, $\mathcal{J}\mu = \mathcal{J}\rho = \rho$ and then $\mathcal{J}\mu = g\rho = \rho$. Hence the result. □

Definition 2.9. Let $(\mathcal{Q}, \Omega, \preceq)$ be a partially ordered metric type space and g is self-map on it. A map $\mathcal{J} : \mathcal{Q} \times \mathcal{Q} \to \mathcal{Q}$ is known to be a generalized $(\hat{\phi}, \hat{\psi}, \hat{\theta})$-contraction w.r.t. g, if

$$\phi(\jmath^k\Omega(\mathcal{J}(\wp,\varpi),\mathcal{J}(\varrho,\sigma))) \le \hat{\phi}(C_g(\wp,\varpi,\varrho,\sigma)) - \hat{\psi}(C_g(\wp,\varpi,\varrho,\sigma)) + \mathcal{M}\hat{\theta}(D_g(\wp,\varpi,\varrho,\sigma)), \tag{51}$$

for $\wp,\varpi,\varrho,\sigma \in \mathcal{Q}$ such that $g\wp \le g\varrho$, $g\varpi \ge g\sigma$, $\ell > 2$, $\jmath > 1$, $\mathcal{M} \ge 0$, $\hat{\phi} \in \hat{\Phi}$, $\hat{\psi} \in \hat{\Psi}$ and $\hat{\theta} \in \Theta$ and, where

$$C_g(\wp,\varpi,\varrho,\sigma) = \max\Big\{\frac{\Omega(g\varrho,\mathcal{J}(\varrho,\sigma))\left[1+\Omega(g\wp,\mathcal{J}(\wp,\varpi))\right]}{1+\Omega(g\wp,g\varpi)}, \frac{\Omega(g\wp,\mathcal{J}(\wp,\varpi))\,\Omega(g\wp,\mathcal{J}(\varrho,\sigma))}{1+\Omega(g\wp,\mathcal{J}(\varrho,\sigma))+\Omega(g\varpi,\mathcal{J}(\wp,\varpi))}, \Omega(g\wp,g\varpi)\Big\}, \tag{52}$$

and

$$D_g(\wp,\varpi,\varrho,\sigma) = \min\{\Omega(g\wp,\mathcal{J}(\wp,\varpi)),\Omega(g\varrho,\mathcal{J}(\varrho,\sigma)),\Omega(g\varpi,\mathcal{J}(\wp,\varpi)),\Omega(g\wp,\mathcal{J}(\varrho,\sigma))\}. \tag{53}$$

Theorem 2.10. *The mappings $\mathcal{J}$ and g defined in Definition 2.9 have a coupled coincidence point if*

(i). $\mathcal{J}$ and g are continuous,
(ii). $\mathcal{J}$ is mixed g-monotone and commutes with g and
(iii). there exists $(\wp_0,\varpi_0) \in \mathcal{Q}\times\mathcal{Q}$ such that $g\wp_0 \le \mathcal{J}(\wp_0,\varpi_0)$, $g\varpi_0 \ge \mathcal{J}(\varpi_0,\wp_0)$ and $\mathcal{J}(\mathcal{Q}\times\mathcal{Q}) \subseteq g(\mathcal{Q})$, where $\mathcal{Q}$ is complete.

Proof. As by Theorem 2.2 [8], there are two sequences $\{\wp_n\},\{\varpi_n\} \subset \mathcal{Q}$ with

$$g\wp_{n+1} = \mathcal{J}(\wp_n,\varpi_n), \quad g\varpi_{n+1} = \mathcal{J}(\varpi_n,\wp_n), \ (n \ge 0), \tag{54}$$

where $\{g\wp_n\}$ is nondecreasing and $\{g\varpi_n\}$ is nonincreasing in $\mathcal{Q}$. Let $\wp = \wp_n, \varpi = \varpi_n, \varrho = \wp_{n+1}, \sigma = \varpi_{n+1}$ in (51), we have

$$\begin{aligned}\hat{\phi}(\jmath^\ell\Omega(g\wp_{n+1},g\wp_{n+2})) &= \hat{\phi}(\jmath^\ell\Omega(\mathcal{J}(\wp_n,\varpi_n),\mathcal{J}(\wp_{n+1},\varpi_{n+1})))\\ &\le \hat{\phi}(C_g(\wp_n,\varpi_n,\wp_{n+1},\varpi_{n+1})) - \hat{\psi}(C_g(\wp_n,\varpi_n,\wp_{n+1},\varpi_{n+1})) + \mathcal{M}\hat{\theta}(D_g(\wp_n,\varpi_n,\wp_{n+1},\varpi_{n+1})),\end{aligned} \tag{55}$$

here

$$C_g(\wp_n,\varpi_n,\wp_{n+1},\varpi_{n+1}) = \max\{\Omega(g\wp_n,g\wp_{n+1}),\Omega(g\wp_{n+1},g\wp_{n+2})\}, \tag{56}$$

and,

$$\begin{aligned}D_g(\wp_n,\varpi_n,\wp_{n+1},\varpi_{n+1}) = \min\{&\Omega(g\wp_n,g\wp_{n+1}),\Omega(g\wp_{n+1},g\wp_{n+2}),\\ &\Omega(g\wp_{n+1},g\wp_{n+1}),\Omega(g\wp_n,g\wp_{n+2})\} = 0.\end{aligned} \tag{57}$$

Therefore from (55) using (56) and (57), we get

$$\begin{aligned}\hat{\phi}(\jmath^\ell\Omega(g\wp_{n+1},g\wp_{n+2})) \le\ &\hat{\phi}(\max\{\Omega(g\wp_n,g\wp_{n+1}),\Omega(g\wp_{n+1},g\wp_{n+2})\})\\ &-\hat{\psi}(\max\{\Omega(g\wp_n,g\wp_{n+1}),\Omega(g\wp_{n+1},g\wp_{n+2})\}).\end{aligned} \tag{58}$$

Also by letting $\wp = \varpi_{n+1}, \varpi = \wp_{n+1}, \varrho = \wp_n$ and, $\sigma = \wp_n$ in (51), we get

$$\begin{aligned}\hat{\phi}(\jmath^\ell\Omega(g\varpi_{n+1},g\varpi_{n+2})) \le\ &\hat{\phi}(\max\{\Omega(g\varpi_n,g\varpi_{n+1}),\Omega(g\varpi_{n+1},g\varpi_{n+2})\})\\ &-\hat{\psi}(\max\{\Omega(g\varpi_n,g\varpi_{n+1}),\Omega(g\varpi_{n+1},g\varpi_{n+2})\}).\end{aligned} \tag{59}$$

It is known that $\max\{\hat{\phi}(\eta_1),\hat{\phi}(\eta_2)\} = \hat{\phi}\{\max\{\eta_1,\eta_2\}\}$ for $\eta_1,\eta_2 \in [0,+\infty)$. Then by adding the equations (58) and (59) together to get,

$$\begin{aligned}\hat{\phi}(\jmath^\ell\Pi_n) \le\ &\phi(\max\{\Omega(g\wp_n,g\wp_{n+1}),\Omega(g\wp_{n+1},g\wp_{n+2}),\Omega(g\varpi_n,g\varpi_{n+1}),\Omega(g\varpi_{n+1},g\varpi_{n+2})\})\\ &-\hat{\psi}(\max\{\Omega(g\wp_n,g\wp_{n+1}),\Omega(g\wp_{n+1},g\wp_{n+2}),\Omega(g\varpi_n,g\varpi_{n+1}),\Omega(g\varpi_{n+1},g\varpi_{n+2})\})\end{aligned} \tag{60}$$

here

$$\Pi_n = \max\{\Omega(g\wp_{n+1},g\wp_{n+2}),\Omega(g\varpi_{n+1},g\varpi_{n+2})\}. \tag{61}$$

Let,

$$\Xi_n = \max\{\Omega(g\wp_n,g\wp_{n+1}),\Omega(g\wp_{n+1},g\wp_{n+2}),\Omega(g\varpi_n,g\varpi_{n+1}),\Omega(g\varpi_{n+1},g\varpi_{n+2})\}, \tag{62}$$

hence from equations (58)-(61), we obtain

$$\jmath^\ell\Pi_n \le \Xi_n. \tag{63}$$

Now to claim that

$$\Pi_n \le \lambda\Pi_{n-1}, \ (n \ge 1), \tag{64}$$

here $0 \le \lambda < \frac{1}{\jmath^\ell}$.

Since $\jmath < 1$ and if $\Xi_n = \Pi_n$ then $\Pi_n = 0$ from (63) and thus (64) holds. If $\Xi_n = \max\{\Omega(g\wp_n,g\wp_{n+1}),\Omega(g\varpi_n,g\varpi_{n+1})\}$, that is, $\Xi_n = \Pi_{n-1}$ then from (63), we get $\Pi_n \le \lambda^n\Pi_0$ and implies that

$$\Omega(g\wp_{n+1},g\wp_{n+2}) \le \lambda^n\Pi_0 \ \text{ and, } \ \Omega(g\varpi_{n+1},g\varpi_{n+2}) \le \lambda^n\Pi_0, \tag{65}$$

from Lemma 3.1 [28] that $\{g p_n\}$ and $\{g \varpi_n\}$ are Cauchy sequences in $\mathcal{Q}$. Thus, $\mathcal{J}$ and g have a coincidence point from Theorem 2.2 [4]. □

Corollary 2.11. *A continuous and mixed monotone mapping $\mathcal{J} : \mathcal{Q} \times \mathcal{Q} \to \mathcal{Q}$ has a coupled fixed point in $\mathcal{Q}$, where $\mathcal{Q}$ is complete for $(p_0, \varpi_0) \in \mathcal{Q} \times \mathcal{Q}$ with $p_0 \preceq \mathcal{J}(p_0, \varpi_0)$ and $\varpi_0 \succeq \mathcal{J}(\varpi_0, p_0)$ satisfying the following contractions:*

(i).

$$\hat{\phi}(s^{\ell} \Omega(\mathcal{J}(p, \varpi), \mathcal{J}(\varrho, \sigma))) \le \hat{\phi}(C_g(p, \varpi, \varrho, \sigma)) - \hat{\psi}(C_g(p, \varpi, \varrho, \sigma)) + \mathcal{M}\hat{\theta}(D_g(p, \varpi, \varrho, \sigma)), \tag{66}$$

(ii).

$$\Omega(\mathcal{J}(p, \varpi), \mathcal{J}(\varrho, \sigma)) \le \frac{1}{s^{\ell}} C_g(p, \varpi, \varrho, \sigma) - \frac{1}{s^{\ell}} \hat{\psi}(C_g(p, \varpi, \varrho, \sigma)), \ \mathcal{M} = 0, \tag{67}$$

where

$$C_g(p, \varpi, \varrho, \sigma) = \max\{ \frac{\Omega(\varrho, \mathcal{J}(\varrho, \sigma)) \left[1 + \Omega(p, \mathcal{J}(p, \varpi))\right]}{1 + \Omega(p, \varpi)}, \frac{\Omega(p, \mathcal{J}(p, \varpi))\, \Omega(p, \mathcal{J}(\varrho, \sigma))}{1 + \Omega(p, \mathcal{J}(\varrho, \sigma)) + \Omega(\varpi, \mathcal{J}(p, \varpi))}, \Omega(p, \varpi)\}, \tag{68}$$

and

$$D_g(p, \varpi, \varrho, \sigma) = \min\{\Omega(p, \mathcal{J}(p, \varpi)), \Omega(\varrho, \mathcal{J}(\varrho, \sigma)), \Omega(\varpi, \mathcal{J}(p, \varpi)), \Omega(p, \mathcal{J}(\varrho, \sigma))\}, \tag{69}$$

for $p, \varpi, \varrho, \sigma \in \mathcal{Q}$ with $p \preceq \varrho$ and $\varpi \succeq \sigma$, $\ell > 2$, $s > 1$ and, $\hat{\phi} \in \hat{\Phi}$, $\hat{\psi} \in \hat{\Psi}$ and $\hat{\theta} \in \Theta$.

Theorem 2.12. *According to Theorem 2.10, $\mathcal{J}$ and g have a unique coupled common fixed point, if for all $(\gamma^*, \eta^*) \in \mathcal{Q} \times \mathcal{Q}$ so that $(\mathcal{J}(\gamma^*, \eta^*), \mathcal{J}(\eta^*, \gamma^*))$ is comparable to both $(\mathcal{J}(p, \varpi), \mathcal{J}(\varpi, p))$ and $(\mathcal{J}(k, m), \mathcal{J}(m, k))$.*

Proof. A coupled coincidence point for $\mathcal{J}$ and g exists by Theorem 2.10. For uniqueness, let $(p, \varpi), (k, m)$ be two coupled coincidence points of $\mathcal{J}$ and g, whence to show that $g p = g k$ and $g \varpi = g m$. By the hypotheses, if for some $(\gamma^*, \eta^*) \in \mathcal{Q} \times \mathcal{Q}$, $(\mathcal{J}(\gamma^*, \eta^*), \mathcal{J}(\eta^*, \gamma^*))$ is comparable to $(\mathcal{J}(p, \varpi), \mathcal{J}(\varpi, p))$.

Suppose that,

$$\begin{aligned} &(\mathcal{J}(p, \varpi), \mathcal{J}(\varpi, p)) \le (\mathcal{J}(\gamma^*, \eta^*), \mathcal{J}(\eta^*, \gamma^*)) \text{ and} \\ &(\mathcal{J}(k, m), \mathcal{J}(m, k)) \le (\mathcal{J}(\gamma^*, \eta^*), \mathcal{J}(\eta^*, \gamma^*)). \end{aligned} \tag{70}$$

Let ${\gamma^*}_0 = \gamma^*$ and ${\eta^*}_0 = \eta^*$ and then a point $({\gamma^*}_1, {\eta^*}_1) \in \mathcal{Q} \times \mathcal{Q}$ so that

$$g{\gamma^*}_1 = \mathcal{J}({\gamma^*}_0, {\eta^*}_0), \quad g{\eta^*}_1 = \mathcal{J}({\eta^*}_0, {\gamma^*}_0), \text{ for } (n \ge 1). \tag{71}$$

Consequently, we have two sequences $\{g{\gamma^*}_n\}$ and $\{g{\eta^*}_n\}$ in $\mathcal{Q}$ by

$$g{\gamma^*}_{n+1} = \mathcal{J}({\gamma^*}_n, {\eta^*}_n), \quad g{\eta^*}_{n+1} = \mathcal{J}({\eta^*}_n, {\gamma^*}_n), \ (n \ge 0). \tag{72}$$

As by similar processor, we obtain the sequences $\{g p_n\}$, $\{g \varpi_n\}$ and $\{g k_n\}$, $\{g \mathcal{J}_n\}$ in $\mathcal{Q}$ by letting $p_0 = p$, $\varpi_0 = \varpi$ and $k_0 = k$, $m_0 = m$ and then

$$g p_n \to \mathcal{J}(p, \varpi), \ g \varpi_n \to \mathcal{J}(\varpi, p), \ g k_n \to \mathcal{J}(k, m), \ g \mathcal{J}_n \to \mathcal{J}(m, k) \ \ (n \ge 1). \tag{73}$$

Inductively, we get

$$(g p_n, g \varpi_n) \le (g{\gamma^*}_n, g{\eta^*}_n), \ n \ge 0. \tag{74}$$

Now from the contraction condition (51), we have

$$\begin{aligned} \hat{\phi}(\Omega(g p, g{\gamma^*}_{n+1})) &\le \hat{\phi}(s^{\ell} \Omega(g p, g{\gamma^*}_{n+1})) = \hat{\phi}(s^{\ell} \Omega(\mathcal{J}(p, \varpi), \mathcal{J}({\gamma^*}_n, {\eta^*}_n))) \\ &\le \hat{\phi}(C_g(p, \varpi, {\gamma^*}_n, {\eta^*}_n)) - \hat{\psi}(C_g(p, \varpi, {\gamma^*}_n, {\eta^*}_n)) \\ &\quad + \mathcal{M}\hat{\theta}(D_g(p, \varpi, {\gamma^*}_n, {\eta^*}_n)), \end{aligned} \tag{75}$$

where

$$\begin{aligned} C_g(p, \varpi, {\gamma^*}_n, {\eta^*}_n) &= \max\{ \frac{\Omega(g{\gamma^*}_n, \mathcal{J}({\gamma^*}_n, {\eta^*}_n)) \left[1 + \Omega(g p, \mathcal{J}(p, \varpi))\right]}{1 + \Omega(g p, g \varpi)}, \\ &\qquad \frac{\Omega(g p, \mathcal{J}(p, \varpi))\, \Omega(g p, \mathcal{J}({\gamma^*}_n, {\eta^*}_n))}{1 + \Omega(g p, \mathcal{J}({\gamma^*}_n, {\eta^*}_n)) + \Omega(g \varpi, \mathcal{J}(p, \varpi))}, \Omega(g p, g{\gamma^*}_n)\} \\ &= \max\{0, \Omega(g p, g{\gamma^*}_n)\} \\ &= \Omega(g p, g{\gamma^*}_n) \end{aligned} \tag{76}$$

and

$$\begin{aligned} D_g(p, \varpi, {\gamma^*}_n, {\eta^*}_n) = \min\{&\Omega(g p, \mathcal{J}(p, \varpi)), \Omega(g{\gamma^*}_n, \mathcal{J}({\gamma^*}_n, {\eta^*}_n)), \Omega(g \varpi, \mathcal{J}(p, \varpi)), \\ &\Omega(g p, \mathcal{J}({\gamma^*}_n, {\eta^*}_n))\} = 0. \end{aligned} \tag{77}$$

The equations (75)-(77) follows that,

$$\hat{\phi}(\Omega(g\, p, g\gamma^*{}_{n+1})) \le \hat{\phi}(\Omega(g\, p, g\gamma^*{}_n)) - \hat{\psi}(\Omega(g\, p, g\gamma^*{}_n)). \tag{78}$$

Also by similar argument, we have

$$\hat{\phi}(\Omega(g\varpi, g\eta^*{}_{n+1})) \le \hat{\phi}(\Omega(g\varpi, g\eta^*{}_n)) - \hat{\psi}(\Omega(g\varpi, g\eta^*{}_n)). \tag{79}$$

Therefore from the equations (75) and (79), we get

$$\begin{aligned} \hat{\phi}(\max\{\Omega(g\, p, g\gamma^*{}_{n+1}), \Omega(g\varpi, g\eta^*{}_{n+1})\}) &\le \hat{\phi}(\max\{\Omega(g\, p, g\gamma^*{}_n), \Omega(g\varpi, g\eta^*{}_n)\}) \\ &\quad - \hat{\psi}(\max\{\Omega(g\, p, g\gamma^*{}_n), \Omega(g\varpi, g\eta^*{}_n)\}) \\ &< \hat{\phi}(\max\{\Omega(g\, p, g\gamma^*{}_n), \Omega(g\varpi, g\eta^*{}_n)\}), \end{aligned} \tag{80}$$

which implies by $\hat{\phi}$ that

$$\max\{\Omega(g\, p, g\gamma^*{}_{n+1}), \Omega(g\varpi, g\eta^*{}_{n+1})\} < \max\{\Omega(g\, p, g\gamma^*{}_n), \Omega(g\varpi, g\eta^*{}_n)\}. \tag{81}$$

From which the sequence, $\max\{\Omega(g\, p, g\gamma^*{}_n), \Omega(g\varpi, g\eta^*{}_n)\}$ is non-increasing and bounded below and hence from known result, we get

$$\lim_{n\to+\infty} \max\{\Omega(g\, p, g\gamma^*{}_n), \Omega(g\varpi, g\eta^*{}_n)\} = \Lambda, \ \Lambda \ge 0. \tag{82}$$

As $n \to +\infty$ in equation (80), we obtain that

$$\hat{\phi}(\Lambda) \le \hat{\phi}(\Lambda) - \hat{\psi}(\Lambda). \tag{83}$$

Consequently, (82) follows that $\hat{\psi}(\Lambda) = 0$ and then $\Lambda = 0$. Thus,

$$\lim_{n\to+\infty} \max\{\Omega(g\, p, g\gamma^*{}_n), \Omega(g\varpi, g\eta^*{}_n)\} = 0, \tag{84}$$

which implies that,

$$\lim_{n\to+\infty} \Omega(g\, p, g\gamma^*{}_n) = 0 \ \text{ and } \lim_{n\to+\infty} \Omega(g\varpi, g\eta^*{}_n) = 0. \tag{85}$$

As by similar process, we have

$$\lim_{n\to+\infty} \Omega(g\, k, g\gamma^*{}_n) = 0 \text{ and } \lim_{n\to+\infty} \Omega(g\mathcal{J}, g\eta^*{}_n) = 0. \tag{86}$$

Therefore from (85) and (86), we get $g\, p = g\, k$ and $g\varpi = g\mathcal{J}$. As $\mathcal{J}$ commutes with g and also $g\, p = \mathcal{J}(p, \varpi)$ and $g\varpi = \mathcal{J}(\varpi, p)$ suggests that

$$g(g\, p) = g(\mathcal{J}(p, \varpi)) = \mathcal{J}(g\, p, g\varpi), \ g(g\varpi) = g(\mathcal{J}(\varpi, p)) = \mathcal{J}(g\varpi, g\, p). \tag{87}$$

If $g\, p = \gamma_1^*$ and $g\varpi = \eta_1^*$ then from (87), we get

$$g(\gamma_1^*) = \mathcal{J}(\gamma_1^*, \eta_1^*) \text{ and } g(\eta_1^*) = \mathcal{J}(\eta_1^*, \gamma_1^*). \tag{88}$$

Thus, $\mathcal{J}$ and g have a coupled coincidence point (γ_1^*, η_1^*) by (88). Which results in $g(\gamma_1^*) = g\, k$ and $g(\eta_1^*) = g\mathcal{J}$. Hence, $g(\gamma_1^*) = \gamma_1^*$ and $g(\eta_1^*) = \eta_1^*$.

Let (γ_2^*, η_2^*) be another coupled common fixed point. Then $\gamma_2^* = g\gamma_2^* = \mathcal{J}(\gamma_2^*, \eta_2^*)$ and $\eta_2^* = g\eta_2^* = \mathcal{J}(\eta_2^*, \gamma_2^*)$. But (γ_2^*, η_2^*) is a coupled common fixed point for $\mathcal{J}$, g then $g\gamma_2^* = g\, p = \gamma_1^*$ and $g\eta_2^* = g\varpi = \eta_1^*$. Therefore, $\gamma_2^* = g\gamma_2^* = g\gamma_1^* = \gamma_1^*$ and $\eta_2^* = g\eta_2^* = g\eta_1^* = \eta_1^*$. Hence the uniqueness. □

Theorem 2.13. *The unique common fixed point for $\mathcal{J}$ and g in Theorem 2.12 exists, if $g\, p_0 \preceq g\varpi_0$ or $g\, p_0 \succeq g\varpi_0$.*

Proof. We have to show that $p = \varpi$ for a unique common fixed point $(p, \varpi) \in \mathcal{Q}$ of the the mappings $\mathcal{J}$ and g. If $g\, p_0 \preceq g\varpi_0$ then $g\, p_n \preceq g\varpi_n$, $(n \ge 0)$ by induction. As by Lemma 2 [29], we have

$$\begin{aligned} \hat{\phi}(s^{\ell-2}\Omega(p, \varpi)) = \hat{\phi}(s^{\ell}\frac{1}{s^2}\Omega(p, \varpi)) &\le \lim_{n\to+\infty}\sup \hat{\phi}(s^{\ell}\Omega(p_{n+1}, \varpi_{n+1})) \\ &= \lim_{n\to+\infty}\sup \hat{\phi}(s^{\ell}\Omega(\mathcal{J}(p_n, \varpi_n), \mathcal{J}(\varpi_n, p_n))) \\ &\le \lim_{n\to+\infty}\sup \hat{\phi}(\mathcal{C}_g(p_n, \varpi_n, \varpi_n, p_n)) - \lim_{n\to+\infty}\inf \hat{\psi}(\mathcal{C}_g(p_n, \varpi_n, \varpi_n, p_n)) \\ &\quad + \lim_{n\to+\infty}\sup \mathcal{M}\hat{\theta}(\mathcal{D}_f(p_n, \varpi_n, \varpi_n, p_n)) \\ &\le \hat{\phi}(\Omega(p, \varpi)) - \lim_{n\to+\infty}\inf \hat{\psi}(\mathcal{C}_f(p_n, \varpi_n, \varpi_n, p_n)) \\ &< \hat{\phi}(\Omega(p, \varpi)), \end{aligned} \tag{89}$$

thus (89) follows a contraction. As a result we get $p = \varpi$.

Also get the same conclusion, if $g\, p_0 \succeq g\varpi_0$. □

Remark 2.14. If $s = 1$, the contraction condition

$$\hat{\phi}(\Omega(\mathcal{J}(p, \varpi), \mathcal{J}(\varrho, \varpi))) \le \hat{\phi}(\max\{\Omega(g\, p, g\varrho), \Omega(g\varpi, g\varpi)\}) - \hat{\psi}(\max\{\Omega(g\, p, g\varrho), \Omega(g\varpi, g\varpi)\}) \tag{90}$$

becomes,

$$\Omega(\mathcal{J}(\wp,\varpi),\mathcal{J}(\varrho,\varpi)) \le \varphi(\max\{\Omega(\mathcal{g}\wp,\mathcal{g}\varrho),\Omega(\mathcal{g}\varpi,\mathcal{g}\varpi)\}), \tag{91}$$

as by the consequence of [27], where $\varphi : [0,+\infty) \to [0,+\infty)$ is a continuous mapping with $\varphi(\eta) < \eta$ for any $\eta > 0$ as well as $\varphi(\eta) = 0$ iff $\eta = 0$ and $\hat{\phi} \in \hat{\Phi}$, $\hat{\psi} \in \hat{\Psi}$. Therefore, the results obtained in this paper are generalizing and extending the findings in [15, 22, 31, 32] and many outcomes in the literature.

Below are some examples given based on a metric Ω is continuous or discontinuous.

Example 2.15. A metric $\Omega : \mathcal{Q} \times \mathcal{Q} \to \mathcal{Q}$, where $\mathcal{Q} = \{a,b,c,d,e,f\}$ defined as

$\Omega(\wp,\varpi) = \Omega(\varpi,\wp) = 0,$ if $\wp = \varpi = a,b,c,d,e,f$ and $\wp = \varpi$,

$\Omega(\wp,\varpi) = \Omega(\varpi,\wp) = 3,\ if\ \wp = \varpi = a,b,c,d,e$ and $\wp \neq \varpi$,

$\Omega(\wp,\varpi) = \Omega(\varpi,\wp) = 12,\ if\ \wp = a,b,c,d$ and $\varpi = f$,

$\Omega(\wp,\varpi) = \Omega(\varpi,\wp) = 20,\ if\ \wp = e$ and $\varpi = f$, with usual order $\le$.

Let $\mathcal{J} : \mathcal{Q} \to \mathcal{Q}$ be a mapping defined as $\mathcal{J}a = \mathcal{J}b = \mathcal{J}c = \mathcal{J}d = \mathcal{J}e = 1, \mathcal{J}f = 2$, $\hat{\phi}(\eta) = \frac{\eta}{2}$ and $\hat{\psi}(\eta) = \frac{\eta}{4}$, for $\eta \in [0,+\infty)$. Then $\mathcal{J}$ has a fixed point in $\mathcal{Q}$.

Proof. The metric is complete for $\mathfrak{s} = 2$ and then $(\mathcal{Q},\Omega,\le)$ is a c.p.o.b-m.s.. Also its clear that Ω is continuous. Let $\wp,\varpi \in \mathcal{Q}$ with $\wp < \varpi$ then,
Case (i). Suppose $\wp,\varpi \in \{a,b,c,d,e\}$, then we have $\Omega(\mathcal{J}\wp,\mathcal{J}\varpi) = \Omega(a,a) = 0$. Therefore

$$\hat{\phi}(2\Omega(\mathcal{J}\wp,\mathcal{J}\varpi)) \le \hat{\phi}(C(\wp,\varpi)) - \hat{\psi}(C(\wp,\varpi)). \tag{92}$$

Case (ii). Suppose $\wp \in \{a,b,c,d,e\}$ and $\varpi = f$, then we have $\Omega(\mathcal{J}\wp,\mathcal{J}\varpi) = \Omega(a,b) = 3$ and $C(f,e) = 20$, $C(\wp,f) = 12$, where $\wp \in \{a,b,c,d\}$. Thus,

$$\hat{\phi}(2\Omega(\mathcal{J}\wp,\mathcal{J}\varpi)) \le \frac{C(\wp,\varpi)}{d} = \hat{\phi}(C(\wp,\varpi)) - \hat{\psi}(C(\wp,\varpi)). \tag{93}$$

Hence the conclusion as all assumptions of Corollary 2.4 are fulfilled. □

Example 2.16. With usual order $\le$, define a metric Ω by

$$\Omega(\wp,\varpi) = \begin{cases} 0 & ,\ for\ \wp = \varpi \\ 1 & ,\ for\ \wp \neq \varpi \in \{0,1\} \\ |\wp - \varpi| & ,\ for\ \wp,\varpi \in \{0, \frac{1}{2n}, \frac{1}{2m} : n \neq m \ge 1\} \\ 2 & ,\ othercases \end{cases} \tag{94}$$

where $\mathcal{Q} = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \frac{1}{n}, \ldots\}$. A mapping $\mathcal{J} : \mathcal{Q} \to \mathcal{Q}$ has a fixed point with $\mathcal{J}0 = 0, \mathcal{J}\frac{1}{n} = \frac{1}{12n}, n \ge 1$ and $\hat{\phi}(\eta) = \eta$, $\hat{\psi}(\eta) = \frac{4\eta}{5}$ where $\eta \in [0,+\infty)$.

Proof. By definition of a metric Ω is discontinuous and for $\mathfrak{s} = \frac{12}{5}$, $(\mathcal{Q},\Omega,\le)$ is a c.p.o.b-m.s. For $\wp,\varpi \in \mathcal{Q}$ such that $\wp < \varpi$, then:
Case (i). Let $\wp = 0$ and $\varpi = \frac{1}{n}$ for $n > 0$. Then $\Omega(\mathcal{J}\wp,\mathcal{J}\varpi) = \frac{1}{12n}$ and $C(\wp,\varpi) = \frac{1}{n}$ and $C(\wp,\varpi) = \{1,2\}$. Thus,

$$\hat{\phi}\Big(\frac{12}{5}\Omega(\mathcal{J}\wp,\mathcal{J}\varpi)\Big) \le \frac{C(\wp,\varpi)}{5} = \hat{\phi}(C(\wp,\varpi)) - \hat{\psi}(C(\wp,\varpi)). \tag{95}$$

Case (ii). Let $\wp = \frac{1}{m}$ and $\varpi = \frac{1}{n}$ where $m > n \ge 1$, then

$$\Omega(\mathcal{J}\wp,\mathcal{J}\varpi) = \Omega(\frac{1}{12m},\frac{1}{12n}),\ C(\wp,\varpi) \ge \frac{1}{n} - \frac{1}{m} \text{ or } C(\wp,\varpi) = 2. \tag{96}$$

Therefore,

$$\hat{\phi}\Big(\frac{12}{5}\Omega(\mathcal{J}\wp,\mathcal{J}\varpi)\Big) \le \frac{C(\wp,\varpi)}{5} = \hat{\phi}(C(\wp,\varpi)) - \hat{\psi}(C(\wp,\varpi)). \tag{97}$$

Therefore, by Corollary 2.4, we have the result. □

Example 2.17. Let $\mathcal{Q} = \{\mho/\mho : [a_1,a_2] \to [a_1,a_2]$ is continuous map$\}$ and, a metric $\Omega : \mathcal{Q} \times \mathcal{Q} \to \mathcal{Q}$ defined by

$$\Omega(\mho_1,\mho_2) = \sup_{\eta\in[a_1,a_2]} \{|\mho_1(\eta) - \mho_2(\eta)|^2\}$$

for every $\mho_1,\mho_2 \in \mathcal{Q}$, $0 \le a_1 < a_2$ and if $\mho_1 \le \mho_2$ then $a_1 \le \mho_1(\eta) \le \mho_2(\eta) \le a_2, \eta \in [a_1,a_2]$. A mapping $\mathcal{J} : \mathcal{Q} \to \mathcal{Q}$ has a unique fixed point with $\mathcal{J}\mho = \frac{\mho}{5}, \mho \in \mathcal{Q}$ and $\hat{\phi}(\ddot{a}) = \ddot{a}$, $\hat{\psi}(\ddot{a}) = \frac{\ddot{a}}{3}$, $\ddot{a} \in [0,+\infty)$.

Proof. For $\mathfrak{s} = 2$ with above metric all assumptions in Theorem 2.3 are satisfied as $\min(\mho_1,\mho_2)(\eta) = \min\{\mho_1(\eta),\mho_2(\eta)\}$ is continuous. And hence, $0 \in \mathcal{Q}$ is the only fixed point to $\mathcal{J}$. Hence the uniqueness. □

3. Conclusion and future works

The current study explored fixed point results of the mappings satisfies generalized weak contractions in a complete partially ordered b-metric space. The obtained outcomes generalized and, extended some main results in the literature. Further, suitable examples are presented to support the findings in this work.

These results can be extended further by assuming suitable topological properties on mappings as well as various ordered metric spaces like Gb-metric spaces, extended fuzzy cone b-metric space, etc.

Declarations

Author contribution statement

N. Seshagiri Rao: Conceived and designed the analysis; Analyzed and interpreted the data; Contributed analysis tools or data; Wrote the paper

Funding statement

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Data availability statement

No data was used for the research described in the article.

Declaration of interest's statement

The authors declare no conflict of interest.

Additional information

No additional information is available for this paper.

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